Chapter Eight

Square Roots

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The Quadratic Formula

Classification of Numbers

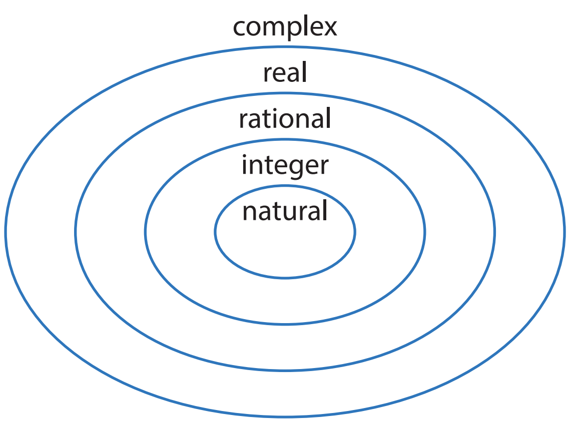
All numbers fall into one or more of the following five categories.

Set	Symbol	Definition	Punctuation	Examples
Natural	$\mathbb{N}$	counting numbers 1, 2, 3, . . .	none	4, 5, 8000
Integers	$\mathbb{Z}$	$\mathbb{N}$, -$\mathbb{N}$, and 0	-	all of $\mathbb{N}$, -4, -8000, 0
Rational	$\mathbb{Q}$	$\mathbb{Z} \div \mathbb{Z}$	- . /	all of $\mathbb{Z}$, $-\frac{1}{2}$, 2.309, $\frac{2309}{10000}$
Real	$\mathbb{R}$	any number on the number line	any	all of $\mathbb{Q}$, $\sqrt{2}$, $\sqrt{3}$, π
Complex	$\mathbb{C}$	any number, including imaginary numbers	any	all of $\mathbb{R}$, $\sqrt{-2}$, $\sqrt{-3}$, $2+i$

Irrational numbers are real numbers that are not rational.

Imaginary numbers are complex numbers that are not real.

Each number belongs to the set defining it and every set including that set, as shown at right. For example, all rational numbers are also real and complex.



Radicals

A **radicand** is an expression under a square root symbol.

A **radical** is a square root symbol and its radicand.

The **conjugate** of $a + \sqrt{b}$ is $a - \sqrt{b}$. $a + \sqrt{b}$ multiplied by its conjugate is the rational number $a^2 - b$, because the irrational terms will cancel.

Operation	Approach	Calculation	Example
Addition	Combine like terms.	$a\sqrt{x} + b\sqrt{x} = (a + b)\sqrt{x}$	$(6 + 8\sqrt{2}) - (5\sqrt{2} + 4\sqrt{3})$ $= 6 + 3\sqrt{2} - 4\sqrt{3}$
Multiplication	Multiply the radicands.	$\sqrt{x}\sqrt{y} = \sqrt{xy}$	$\sqrt{100}\sqrt{3}$ $= \sqrt{300}$
Simplification	Split the radicand into two factors, one of which is square.	$\sqrt{x^2y} = \sqrt{x^2}\sqrt{y} = x\sqrt{y}$	$\sqrt{300}$ $= \sqrt{100}\sqrt{3}$ $= 10\sqrt{3}$
Division	Multiply the numerator and denominator by the denominator's conjugate.	$\frac{x}{y + \sqrt{z}} \cdot \frac{y - \sqrt{z}}{y - \sqrt{z}} = \frac{xy - x\sqrt{z}}{y^2 - z}$	$\frac{10}{3 + \sqrt{5}} \cdot \frac{3 - \sqrt{5}}{3 - \sqrt{5}} = \frac{30 - 10\sqrt{5}}{3^2 - 5}$ $= \frac{30 - 10\sqrt{5}}{4} = \frac{15 - 5\sqrt{5}}{2}$

Complex Arithmetic

The **complex conjugate** of $a + bi$ is $a - bi$. $a + bi$ multiplied by its complex conjugate is the real number $a^2 + b^2$, because the imaginary parts will cancel.

Operation	Approach	Calculation	Example
Addition	Add real parts, and add imaginary parts.	$(a + bi) + (c + di) =$ $a + c + (b + d)i$	$(3 + 4i) + (6 + 10i)$ $= (3 + 6) + (4 + 10)i$ $= 9 + 14i$
Multiplication	Treat i like a variable, but change any instance of i^2 to -1.	$ai \bullet bi = abi^2 = ab(-1)$ $= -ab$	$(3 + 2i)(1 - 4i)$ $= 3 - 12i + 2i - 8i^2$ $= 3 - 10i - 8(-1)$ $= 11 - 10i$
Simplification	Factor out $i = \sqrt{-1}$.	$\sqrt{-x} = \sqrt{-1}\sqrt{x} = i\sqrt{x}$	$\sqrt{-6}$ $= \sqrt{-1}\sqrt{6}$ $= i\sqrt{6}$
Division	Multiply the numerator and denominator by the denominator's conjugate.	$\frac{x}{y + zi} \bullet \frac{y - zi}{y - zi}$ $= \frac{xy - xzi}{y^2 - z^2i^2} = \frac{xy - xzi}{y^2 + z^2}$	$\frac{10}{3 + 5i} \bullet \frac{3 - 5i}{3 - 5i} = \frac{30 - 50i}{3^2 - 25i^2}$ $= \frac{30 - 50i}{34}$

Completing the Square

A perfect square trinomial can be written as a square, making it easy to take the square root.

For $x^2 + bx + c$ to be a square, c must be $c = (\frac{b}{2})^2$. If it is not, the you can **complete the square** by adding (on each side) the amount needed to make it a square. The square will be $(x + \frac{b}{2})^2$.

In the examples below with $b = 20$, c must be $(\frac{20}{2})^2 = 100$ to make it a square.

Equation	To make it a square	Work	Solutions
$x^2 + 20x + 100 = 0$	This is already a square.	$x^2 + 20x + 100 = 0$ $(x + 10)^2 = 0$ $x + 10 = 0$	$x = -10$
$x^2 + 20x + 96 = 0$	Add 4 to each side.	$x^2 + 20x + 100 = 4$ $(x + 10)^2 = 4$ $x + 10 = \pm 2$	$x = -12, x = -8$
$x^2 + 20x + 97 = 0$	Add 3 to each side.	$x^2 + 20x + 100 = 3$ $(x + 10)^2 = 3$ $x + 10 = \pm\sqrt{3}$	$x = -10 \pm \sqrt{3}$
$x^2 + 20x + 104 = 0$	Subtract 4 from each side.	$x^2 + 20x + 100 = -4$ $(x + 10)^2 = -4$ $x + 10 = \pm 2i$	$x = -10 \pm 2i$

For $ax^2 + bx + c$, each term can first be divided by a, but it is usually easier to use a different method.

The Quadratic Formula

Solving the equation $ax^2 + bx + c = 0$ by completing the square yields the solutions $\mathbf{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$. This is called the **quadratic formula**, and it can be used to solve any quadratic equation in standard form.

The radicand $\mathbf{b^2 - 4ac}$ is called the **discriminant**. Be sure to use parentheses around negative numbers.

Equation	Discriminant	Equation	Solutions
$\mathbf{x^2 + 8x + 16 = 0}$	$8^2 - 4(1)(16) = 0$	$x = \frac{-8 \pm \sqrt{0}}{2(1)}$	$x = \frac{-8 \pm 0}{2} = -4$
$\mathbf{x^2 + 8x + 12 = 0}$	$8^2 - 4(1)(12) = 16$	$x = \frac{-8 \pm \sqrt{16}}{2(1)}$	$x = \frac{-8 \pm 4}{2} = -2, -6$
$\mathbf{x^2 - 8x - 12 = 0}$	$(-8)^2 - 4(1)(-12) = 112$	$x = \frac{-8 \pm \sqrt{112}}{2(1)}$	$x = \frac{-8 \pm 4\sqrt{7}}{2} = -4 \pm 2\sqrt{7}$
$\mathbf{x^2 - 8x + 20 = 0}$	$(-8)^2 - 4(1)(20) = -16$	$x = \frac{-8 \pm \sqrt{-16}}{2(1)}$	$x = \frac{-8 \pm 4i}{2} = -4 \pm 2i$

The discriminant indicates whether the solutions will be real or imaginary. Since imaginary x-intercepts don't exist, it also indicates how many x-intercepts the graph has.

Discriminant	Solutions	# of *x*-intercepts	Factorable
positive	two real	two	if discriminant is square
zero	one real	one	as a perfect square
negative	two imaginary	none	no