CHAPTER EIGHT: SQUARE ROOTS

Review April 3 ⊱ Test April 15

Algebraic equations are solved by applying inverse functions. Since quadratics involve squaring values, quadratics can be solved by taking square roots. Every number except zero has two square roots, which are the same as each other except one is positive and one is negative. The square roots of a square number are two whole numbers. The square roots of a fraction of squares are fractions. The square roots of any other positive number are irrational numbers that cannot be exactly written as a decimal. The square roots of a negative number are called imaginary numbers, and they are not on the real number line.

8-A Classification of Numbers — Monday • 3/23

integer • rational • irrational

❶ Identify a number as natural, integer, rational, real, or complex.

8-B Radicals — Wednesday • 3/25

radical • radicand

❶ Add and subtract radical expressions.
❷ Multiply radical expressions.
❸ Simplify the square root of a positive integer.
❹ Simplify the square root of an algebraic term.
❺ Rationalize a denominator.

8-C Complex Numbers — Friday • 3/27

i • real number • imaginary number • complex number • standard form • complex plane • complex conjugate

❶ Plot numbers on the complex plane.
❷ Add and subtract complex numbers.
❸ Simplify the square root of a negative integer.
❹ Multiply complex numbers.

8-D Completing the Square — Monday • 3/30

❶ Solve $x^2 + bx + c = 0$ by completing the square.

8-E The Quadratic Formula — Wednesday • 4/1

quadratic formula • discriminant

❶ Solve a quadratic equation by using the quadratic formula.
❷ Use a discriminant to determine the type of solutions of a quadratic equation.
❸ Use a discriminant to determine the number of x-intercepts of a quadratic function.

8-A Classification of Numbers

Every number is contained in one or more of five number sets.
NATURAL Numbers ($\mathbb{N}$) are counting numbers such as 1, 2, 3, and 2000. They don't require any symbols to write them.
INTEGERS ($\mathbb{Z}$) are the natural numbers and the negative versions of them, and zero.
RATIONAL Numbers ($\mathbb{Q}$) are any integer divided by an integer (other than zero), meaning they are any number that could be written as a fraction of integers.
REAL Numbers ($\mathbb{R}$) are all integers and all numbers inbetween. For example $\sqrt{3}$ is between 1 and 2, and π is between 3 and 4.
COMPLEX Numbers ($\mathbb{C}$) are all numbers, even those involving the square root of a negative such as $\sqrt{-1}$ or $2 + \sqrt{-5}$.
Each set includes the sets above it. For example, all rational numbers are also real and complex.
Numbers that are real but not rational are called IRRATIONAL.
Numbers that are complex but not real are called IMAGINARY.
Subsets of these sets that contain only positive or only negative numbers are denoted with a superscript + or -, such as $\mathbb{R}^+$.

❶ Identify a number as natural, integer, rational, real, or complex.

1. All numbers are complex.
2. If it does not involve an even root of a negative number, it is also real. Otherwise it is imaginary.
3. If it can be written exactly as a fraction of integers, it is also rational. Otherwise it is irrational.
   Keep in mind that all decimals can be written as fractions except those with no pattern or end. For example, $0.27 = \frac{27}{100}$ and $0.272727... = \frac{3}{11}$.
4. If it can be written with no decimal point, fraction bar, or any other symbol other than a negative sign, it is also an integer.
5. If it is a positive integer, it is natural.

❶ Sort the following numbers into the smallest set that includes each: 2, -2, $\sqrt{2}$, $\sqrt{-2}$, -22, 2.2, $2 - \sqrt{-2}$, 2×10^{22}, 2×10^{-22}, π.

1. $\sqrt{-2}$ and $2 - \sqrt{-2}$ involve even (in this case square) roots of negative numbers. They are complex but not real.
2. $\pi \approx 3.1416$ and $\sqrt{2} \approx 1.4142$ are not imaginary, but they cannot be written as exact decimals. They are real but not rational.
3. 2.2 and 2×10^{-22} can be written as exact fractions, but they are not whole numbers. They are rational but not integers.
4. -2 and -22 can be written without fractions, decimals, radicals, etc., but they are not positive. They are integers but not natural.
5. 2 and 2×10^{22} are positive whole numbers. They are natural.

8-B Radicals

An expression under a root sign is a RADICAND. Together with the root sign it is called a RADICAL.

Radical expressions can be simplified by combining like terms. Any values multiplied by the same radical, such as $5\sqrt{3}$ and $-9\sqrt{3}$, are like terms.

❶ Add and subtract radical expressions.

1. Distribute any coefficient, including negatives.
2. Combine like terms.

❶ $(3 + 5\sqrt{7}) - 2(8\sqrt{2} + 3\sqrt{2} - 11\sqrt{7})$

1. $3 + 5\sqrt{7} - 16\sqrt{2} - 6\sqrt{2} + 22\sqrt{7}$
2. $3 + (5\sqrt{7} + 22\sqrt{7}) + (-16\sqrt{2} - 6\sqrt{2})$

   $3 + 27\sqrt{7} - 22\sqrt{2}$

For positive numbers *a* and *b*, $\sqrt{a} \bullet \sqrt{b} = \sqrt{ab}$.

❷ Multiply radical expressions.

1. Distribute.
2. Multiply the radicands together.

❷

a) $\sqrt{6} \bullet \sqrt{3}$

1.

2. $\sqrt{18}$

b) $\sqrt{6}(-3 + \sqrt{3} - 5\sqrt{11})$

$-3\sqrt{6} + \sqrt{6} \bullet \sqrt{3} - \sqrt{6} \bullet 5\sqrt{11}$

$-3\sqrt{6} + \sqrt{18} - 5\sqrt{66}$

Just as $\sqrt{a}$ and $\sqrt{b}$ can be combined (by multiplying) into $\sqrt{ab}$, $\sqrt{ab}$ can be split into $\sqrt{a}\sqrt{b}$. Doing so with one of the values being a square allows that part of the radical to be simplified, such as $\sqrt{12}$ being split into $\sqrt{4}\sqrt{3}$ which is $2\sqrt{3}$.

❸ Simplify the square root of a positive integer.

1. Factor the radical into two factors, the first of which is a square.
2. Take the square root of the square factor, and leave the other factor in the radical.
3. Repeat these steps if there is another square factor in the radical.

❸ $\sqrt{800}$

1. $\sqrt{100}\sqrt{8}$
2. $10\sqrt{8}$
3. $10\sqrt{4}\sqrt{2} = 10 \bullet 2\sqrt{2} = 20\sqrt{2}$

A variable to an even power is a square: $x^{2n} = (x^n)^2$. A variable to an odd power is the variable times a square: $x^{2n+1} = x(x^n)^2$.

❹ Simplify the square root of an algebraic term.

1. For each variable to an odd exponent, drop the exponent by 1 (to make it even), and multiply by the variable (to make up for the dropped 1).
2. Factor the radical into two factors, the first of which is a square. Include all even-exponent variables in the square.
3. Take the square root of the square factor, and leave the other factor in the radical. To take the square root of a variable with an exponent, divide the exponent by 2.
4. Repeat these steps if there is another square factor in the radical.

❹ $\sqrt{50a^{20}b^{21}c^{49}d}$

1. $\sqrt{50a^{20}b^{20}bc^{48}cd}$
2. $\sqrt{25a^{20}b^{20}c^{48}}\sqrt{2bcd}$
3. $5a^{10}b^{10}c^{24}\sqrt{2bcd}$

The CONJUGATE of a number $a + \sqrt{b}$ is $a - \sqrt{b}$.

Multiplying a number by its conjugate results in a rational (nonradical) number: $(a + \sqrt{b})\,(a - \sqrt{b}) = a^2 + a\sqrt{b} - a\sqrt{b} - (\sqrt{b})^2 = a^2 - b$.

A denominator with a radical is not considered simplified. To RATIONALIZE a Denominator is to rewrite a fraction so that there is no radical in the denominator.

❺ Rationalize a denominator.

1. Multiply the numerator and denominator by the denominator if it is a single term, or by the conjugate of the denominator if it has two terms.
2. Simplify the radical.
3. Reduce.

❺ Simplify.

a) $\frac{12}{\sqrt{20}}$

1. $\frac{12}{\sqrt{20}} \bullet \frac{\sqrt{20}}{\sqrt{20}} = \frac{12\sqrt{20}}{20}$
2. $\frac{12(2\sqrt{5})}{20} = \frac{24\sqrt{5}}{20}$
3. $\frac{6\sqrt{5}}{5}$

b) $\frac{12}{8 - \sqrt{20}}$

$\frac{12}{8 - \sqrt{20}} \bullet \frac{8 + \sqrt{20}}{8 + \sqrt{20}} = \frac{96 + 12\sqrt{20}}{8^2 - 20}$

$\frac{96 + 12(2\sqrt{5})}{44} = \frac{96 + 24\sqrt{5}}{44}$

$\frac{24 + 6\sqrt{5}}{11}$

8-C Complex Numbers

The square root of negative 1 is defined as $i = \sqrt{-1}$.

A COMPLEX Number is a number that can be written in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. Complex numbers written in the form $a + bi$ are in STANDARD Form.

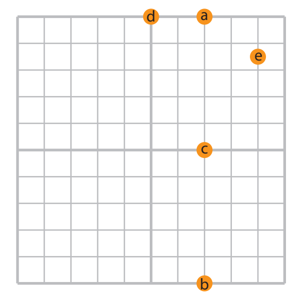
Since a or b can be zero, all numbers are complex. However, the term "complex number" is typically used in the context of imaginary numbers.

Imaginary numbers cannot be plotted on the real number line, but they can be plotted on the COMPLEX PLANE. In this plane, the number $a + bi$ is represented by the point (a, b).

❶ Plot numbers on the complex plane.

1. Write the number as $a + bi$. For real numbers, $a = 0$.
2. Plot the point (a, b).

❶ Plot the following numbers on the complex plane.



a) $2 + 5i$	b) $2 - 5i$	c) 2	d) $5i$	e) $\frac{8+7i}{2}$
1. $2 + 5i$	$2 - 5i$	$2 + 0i$	$0 + 5i$	$4 + \frac{7}{2}i$
$(2, 5)$	$(2, -5)$	$(2, 0)$	$(0, 5)$	$(4, 3.5)$

The real parts of complex numbers are like terms, and the imaginary parts of complex numbers are like terms.

❷ Add and subtract complex numbers.

1. Distribute any coefficient, including negatives.
2. Find the total of the real parts, and find the total of the imaginary parts.

❷ $(3 + 5i) - 4(8 + 3i)$

1. $3 + 5i - 32 - 12i$
2. $-29 - 7i$

The square root of a negative number $-x$ can be written $\sqrt{-x} = \sqrt{x}\,\sqrt{-1} = \sqrt{x}\,i$. If $\sqrt{x}$ is not a whole number, the radical part is written last, after i.

❸ Simplify the square root of a negative integer.

1. Split the number up into a positive number times -1.
2. Take the square root of the positive number normally (see 8-A), and the square root of -1, which is i.

❸ Evaluate.

a) $\sqrt{-25}$

$\sqrt{25}\sqrt{-1} = 5i$

b) $\sqrt{-75}$

$\sqrt{25}\sqrt{-1}\sqrt{3} = 5i\sqrt{3}$

Arithmetically, i functions the same as a variable like x, except that $i^2 = -1$.

❹ Multiply complex numbers.

1. Multiply normally, treating i as a variable.
2. Change every instance of i^2 to -1.

❹ Multiply.

a) $3i \cdot 4i$

1. $12i^2$
2. $-12(-1)$

-12

b) $(2 + 3i)(5 - 6i)$

$10 + 15i - 18i^2$

$10 + 15i - 18(-1)$

$10 + 15i + 18$

$28 + 15i$

c) $(5 + 6i)(5 - 6i)$

$25 - 30i + 30i - 36i^2$

$25 - 30i + 30i - 36(-1)$

$25 + 36$

61

The COMPLEX CONJUGATE of a number $a + bi$ is $a - bi$. A number multiplied by its complex conjugate is a real number, as in the last example above.

8-D Completing the Square

If a perfect square trinomial is set equal to zero, it is easy to solve by first taking the square root of each side of the equation.

If the trinomial is not a perfect square, the same solving method can be used by COMPLETING THE SQUARE, which is adding the amount needed to make it a perfect square and adding the same to the other side of the equation.

For $x^2 + bx + c$ to be a perfect square, c must equal the square of half of b: $c = (\frac{b}{2})^2$. This is very easy to work with if b is an even number.

❶ Solve $x^2 + bx + c = 0$ by completing the square.

1. Find half of b, and square that number to determine what value of c would make a perfect square.
2. Add or subtract on each side whatever value is needed to make the trinomial a perfect square (that is, $c = (\frac{b}{2})^2$). One way to do this is to subtract c from each side and then add $(\frac{b}{2})^2$ to each side.
3. Factor the trinomial. It will factor as the perfect square $(x + \frac{b}{2})^2$.
4. Square root each side.
5. Solve the + equation.
6. Solve the - equation.

❶ $x^2 + 10x + 9 = 0$

1. $(\frac{b}{2})^2 = (\frac{10}{2})^2 = 25$
2. $x^2 + 10x + 25 = -9 + 25$
3. $(x + 5)^2 = 16$
4. $x + 5 = \pm 4$
5. $x + 5 = 4$

   $x = -1$
6. $x + 5 = -4$

   $x = -9$

$ax^2 + bx + c = 0$ can also be solved by completing the square. This can be done by first dividing each term by a and then following the directions above, or by dividing $(\frac{b}{2})^2$ by a when finding the value of c needed to make a perfect square. Either of these approaches will often lead to complicated fractions that make completing the square not an ideal method for solving this type of problem.

8-E The Quadratic Formula

The QUADRATIC FORMULA is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Once put in standard form $(y = ax^2 + bx + c)$, any quadratic equation can be solved by the quadratic formula.

❶ Solve a quadratic equation with the quadratic formula.

1. Put the equation in standard form.
2. Identify a, b, and c.
3. Plug a, b, and c into the quadratic formula.
4. Simplify.

❶ $12x^2 + 17x = 7$

1. $12x^2 + 17x - 7 = 0$
2. $a = 12$, $b = 17$, $c = -7$
3. $x = \frac{-17 \pm \sqrt{17^2 - 4(12)(-7)}}{2(12)}$
4. $x = \frac{-17 \pm 25}{24} = \frac{1}{3}$ or $\frac{-7}{4}$

THE DISCRIMINANT of a Quadratic Polynomial in standard form is $b^2 - 4ac$, which is the radicand of the quadratic formula. It determines what type of solutions a quadradic equation has.

❷ Use a discriminant to identify the type of solutions of a quadratic equation.

1. Put the equation in standard form $ax^2 + bx + c = 0$.
2. Evaluate the discriminant $b^2 - 4ac$.
3. If the discriminant is a square number, then the solutions is rational.
   If the discriminant is a positive number, then the solutions is irrational (but still real).
   If the discriminant is negative, then the solutions is imaginary.
   Because of the ± before the discriminant, there are two solutions to all quadratic equations except if the discriminant is zero.

❷ $2x^2 = 4x - 5$

1. $2x^2 - 4x + 5 = 0$
2. The discriminant is $(-4)^2 - 4(2)(5) = -24$
3. -24 is negative, so there are two imaginary solutions.

There is an x-intercept at each real solution to $ax^2 + bx + c$.

❸ Use a discriminant to identify the number of x-intercepts of a quadratic function.

1. Evaluate the discriminant $b^2 - 4ac$.
2. If the discriminant is positive, there are two x-intercepts.
   If the discriminant is zero, there is one x-intercept.
   If the discriminant is negative, there are no x-intercepts.

❸ $f(x) = 2x^2 - 4x + 5$

1. The discriminant is $(-4)^2 - 4(2)(5) = -24$
2. -24 is negative, so there are two imaginary solutions.