CHAPTER FOUR: PROBABILITY

Review December 1 ✎ Test December 3

Many people find probability to be difficult because it involves more reasoning than straight-forward procedures. For the most part, however, it can be distilled to two fundamental principles: Mathematically, a probability is a fraction with a denominator showing the number of possible equally likely outcomes and a numerator showing how many of those equally likely events are considered successes; and conceptually, probability is based on what information is known, regardless of what has happened or not happened.

4-A Counting Methods — Thursday • 11/12

multiplication principle • permutation • combination • choose • sample space

1. Use a calculator to count combinations.
2. Find the total number of possible outcomes in a series of events.
3. Find the size of a sample space by using permutations, if possible.

4-B Probabilities — Tuesday • 11/17

set • element • cardinality • intersection • union • complement • universal set • empty set • mutually exclusive • given • conditional probability • dependent events • independent events

1. Read set notation.
2. Use the size of a sample space to find the probability of an event.
3. Find the probability of either of two events.
4. Find probabilities based on given information.
5. Identify whether events are dependent or independent.
6. Calculate the probability of multiple events.
7. Calculate the probability of an event that can occur in different ways.
8. Calculate the probability of an event that can occur in different orders.
9. Calculate the probability of at least or at most x out of n occurrences of an event.

4-C Expected Value — Monday • 11/30

expected value

1. Calculate the expected value of a casino event.

4-A Counting Methods

A COMBINATION is a group of items not assigned specific labels or placements within the group. *N* CHOOSE *R*, written $\binom{n}{r}$ or *nCr*, is the number of combinations of *r* items that can be made from a set of *n* items. $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. Note that $\binom{n}{0} = 1$, $\binom{n}{1} = n$, and $\binom{n}{n-r} = \binom{n}{r}$.

❶ Use a calculator to count combinations. (The directions may be different for your calculator.)

1. Type the value of *n*, which is the total number of items, places, etc. there are to choose from.
2. Push [MATH] and choose `PRB`.
3. Choose `nCr`.
4. Type the value of *r*, which is the number of items, people, etc., that are being grouped together with a specific label.

❶ State the number of combinations there are for each of the following, and list two possible combinations.

a) 3 out of 10 applicants are selected to win a scholarship.
1. $n = 10$ applicants
4. $r = 3$ winners
`10 nCr 3` = 120 different possible combinations,
such as Analee, Blake, and Carson or Carson, Analee, and Blake

b) 2 coins land on heads and 4 land on tails.
$n = 6$ coins
$r = 2$ heads
`6 nCr 2` = 15 different possible combinations,
such as the last two flips are heads or the first and third flips are heads.

A SAMPLE SPACE consists of all the possible outcomes of an event, such as "♠, ♥, ♦, ♣" for suit of a card, or "♠♠♦, ♠♦♠, ♦♠♠" for a combination of two spades and a diamond.

The size of a sample space can be found using choose. In the examples above, there are $\binom{4}{1} = 4$ ways to choose one of the four suits, and there are $\binom{3}{2} = 3$ ways to choose which 2 out of 3 cards are spades.

The MULTIPLICATION PRINCIPLE states that the total number of possible outcomes of a series of events is the product of the sizes of the individual sample spaces.

❷ Find the total number of possible outcomes in a series of events.

1. Identify the size of the sample space of each individual event, using $\binom{n}{r}$ as needed.
2. Multiple these sizes together.

❷ State the number of possible outcomes of the following.

a) Choose 3 representatives out of 9 seniors and 2 representatives out of 7 juniors.
$\binom{9}{3}\binom{7}{2} = 84 \bullet 21 = 1764$

b) Identify the 1st place, 2nd place, 3rd place, and 4th place finisher out of 25 racers.
$\binom{25}{1}\binom{24}{1}\binom{23}{1}\binom{22}{1} = 25 \bullet 24 \bullet 23 \bullet 22 = 303{,}600$

The outcome of a series of events that each involve choosing one item from those remaining, as in example ❷b above, is called a PERMUTATION. This is the same as a combination, except that each item in the group is assigned a specific label, such as a title or placement, within the group. In other words, a combination is a selection of a group of r items from a group of n items, and a permutation is a selection of r items, *each identified individually*, from a group of n items.

Permutations are never required, but can be used as an easier way to indicate multiple combinations in which one of the remaining elements is chosen each time. For example, $\binom{9}{1}\binom{8}{1}\binom{7}{1}\binom{6}{1}\binom{5}{1}\binom{4}{1}$ can be written as ${}_9P_6$.

The number of permutations of r items that can be made from a set of n items is $nPr = \frac{n!}{(n-r)!}$, which can be calculated using `nPr` on the calculator.

❸ Find the size of a sample space by using permutations, if possible.

1. Express the number of possible outcomes as a series of combinations.
2. If each combination involves removing a single item, that is, $\binom{n}{1}\binom{n-1}{1}\binom{n-2}{1}\binom{n-3}{1}\ldots$, the expression can be rewritten as nPr, where r is the total number of items being chosen. Otherwise, permutations do not apply.

❸ Use combinations to express the size of the sample space for each of the following. Then rewrite the solution using permutations if possible, or explain why not.

a) Kailey chooses her 3 favorite months.
$\binom{12}{3}$ cannot be written as a permutation, because she is choosing a group of three equal items, not three seperate, distinguishable items.

b) Kailey chooses her favorite, second favorite, and third favorite month.
$\binom{12}{1}\binom{11}{1}\binom{10}{1}$ can be written ${}_{12}P_3$.

c) Use a 6-sided die to choose a color for each of three teams.
$\binom{6}{1}\binom{6}{1}\binom{6}{1}$ cannot be written as a permutation, because the colors are not being removed as they are chosen.

d) Put 6 colors in a hat and draw three of them to choose a different color for each of three teams.
$\binom{6}{1}\binom{5}{1}\binom{4}{1}$ can be written ${}_6P_3$.

4-B Probabilities

A SET is a collection of items, called ELEMENTS. $x \in A$ means x is an element of set A.

The CARDINALITY of a Set A, $|A|$, is the number of elements it contains.

The INTERSECTION of Two Sets A and B, $A \cap B$, is the set of elements that are in both A and B.

The UNION of Two Sets A and B, $A \cup B$, is the set of elements that are in either A or B. An element in $A \cup B$ could be just in A, just in B, or in both A and B.

The COMPLEMENT of a Set A , A', is the set of elements not in A.

The UNIVERSAL Set, U, is the set of all elements in the sample space.

The EMPTY Set, $\varnothing$, contains no elements.

❶ Read set notation.

1. Use the definitions above. In general, $\cap$ means *and*, $\cup$ means *or*, ' means *not*, and $|\ \ |$ means *number of*.

❶ State the following in words, given A is the set of aces and B is the set of black cards.

a) A	the aces
b) U	all of the cards
c) $\varnothing$	none of the cards
d) $\lvert A\rvert$	the number of aces
e) $\lvert U\rvert$	the number of cards
f) $A \cup B$	the cards that are an ace or black
g) $A \cap B$	the cards that are an ace and black
h) $\lvert A \cap B\rvert$	the number of black aces
i) A'	the cards that are not aces
j) $(A \cap B)'$	the cards that are not black aces
k) $(A \cup B)'$	the cards that are not an ace or black

In probability, a set is used to represent an event, with the elements in the set being the event's possible outcomes.

The probability of event A, $P(A)$, is the probability of an outcome in A occurring. $P(A) = \frac{|A|}{|U|}$.

Because $|A|$ and $|U|$ have specific meanings, reducing a probability or converting it to a decimal or a percent causes information to be lost. Therefore, do not convert probabilities to decimals or percents in this class, and do not reduce probabilities, except for those equal to 0 or 1.

❷ Use the size of a sample space to find the probability of an event.

1. Identify the denominator $|U|$, the size of the sample space. Use combinations if needed (see 4-A).
2. Identify the numerator $|A|$, which is the number of possible outcomes for event A within the sample space. Use combinations if needed (see 4-A).

❷ Logan draws two cards. Find the probability that…

a) the first card is a queen

1. There are $|U| = 52$ possible cards.
2. There are $|A| = 4$ possible queens.
3. $P(A) = \frac{4}{52}$

b) both cards are queens

There are $|U| = \binom{52}{2} = 1326$ possible combinations of 2 of the 52 cards.

There are $|A| = \binom{4}{2} = 6$ possible combinations of 2 of the 4 queens.

$P(A) = \frac{6}{1326}$

Two events are said to be MUTUALLY EXCLUSIVE, or Disjoint, if the occurrence of one eliminates the possibility of the other, such as clubs and hearts on a single card. If A and B are mutually exclusive, the probability that one of them will happen is $P(A \cup B) = P(A) + P(B)$.

The above formula does not work for events that are not mutually exclusive, because some outcomes would be counted more than once. To account for this, the overlap between the events can be subtracted: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

❸ Find the probability of either of two events.

1. Add the probabilities of the two events.
2. Subtract the probability of the two events happening simultaneously, since this has been double-counted.

❸ Find the probability of a card being as stated.

a) red or an ace

1. $\frac{26}{52} + \frac{4}{52} = \frac{30}{52}$
2. $\frac{30}{52} - \frac{2}{52} = \frac{28}{52}$

b) a queen or an ace

$\frac{4}{52} + \frac{4}{52} = \frac{8}{52}$

$\frac{8}{52} - 0 = \frac{8}{52}$

GIVEN means *known* (or *knowing*) and is indicated with the $|$ symbol. Probability is based on what is known, not on what has happened.

Given information changes the universal set. For example, given a card is red, the probability of it being hearts is $\frac{13}{26}$, because U now only includes the 26 red cards: $P(\text{hearts}) = \frac{13}{52}$, but $P(\text{hearts} \mid \text{red}) = \frac{13}{26}$.

CONDITIONAL Probabilities incorporate given information that would not necessarily be assumed, such as *the card is red*. The probability of event A after knowledge of event B has been taken into account is the probability of A given B: $P(A \mid B) = \frac{|A \cap B|}{|B|}$.

❹ Find probabilities based on given information.

1. Take into account all known (given) information, even for events that have not yet occurred.
2. Do not assume any information, even for events that have already occurred.

❹ Find the following probabilities for Logan's two cards.

a) The second card is an ace.

2. The first card remains unknown. Do not assume it is or is not an ace. There are still $|U| = 52$ possible outcomes, $|A| = 4$ of which are aces, so $P(A) = \frac{4}{52}$.

b) The second card is an ace, given the first card is an ace.

1. The first card is known, so use that information. There are only $|U| = 51$ remaining possible outcomes for the second card, $|A| = 3$ of which are aces, so $P(A) = \frac{3}{51}$.

c) The first card is an ace, given the second card will be an ace.

1. The second card is known (even though it hasn't been flipped yet), so use that information. There are only $|U| = 51$ remaining possible outcomes for the first card, $|A| = 3$ of which are aces, so $P(A) = \frac{3}{51}$.

The probabilities of DEPENDENT Events are conditional: They are influenced by each others' outcomes.
The probabilities of INDEPENDENT Events do not change based on other events' outcomes.

❺ Identify whether events are dependent or independent.

1. If the probability of an event is the same on every trial, the event is independent.
2. If the outcome of one event changes the sample space for another event, these events are dependent.

❺ Ashley rolls three dice, checking for a 6 each time, and she draws two cards, checking for an ace each time.

1. The die rolls are independent: Each die has a $\frac{1}{6}$ chance of rolling a 6, regardless of the previous rolls.
2. The card draws are dependent: After she knows the first card, there are only 51 remaining possibilities for the second card. The second card has a $\frac{4}{51}$ chance of being an ace if the first card was not an ace, or a $\frac{3}{51}$ chance of being an ace if the first card was an ace.

The probability of both A and B occurring is the product of their probabilities: $P(A \cap B) = P(A) \bullet P(B)$. If the events are dependent, the probability of one of them must be adjusted to its conditional probability based on the other event: $P(A \cap B) = P(A) \bullet P(B \mid A)$, or equivalently, $P(B) \bullet P(A \mid B)$.

❻ Calculate the probability of multiple events.

1. List what has to occur. Ignore events that do not matter.
2. Identify the conditional probability of each event, given each of the previous events occurring.
3. Multiply all the probabilities together.

❻ James draws five cards. Calculate the probability that the first two cards are aces and the fourth card is not an ace.

1. ace, ace, not ace
   (Note that the third and fifth card are irrelevant to this problem.)
2. $P(\text{first card is an ace}) = \frac{4}{52}$
   $P(\text{second card is an ace, given first card is an ace}) = \frac{3}{51}$
   $P(\text{fourth card is not an ace, given first two cards are aces}) = \frac{48}{50}$
3. $\frac{4}{52} \bullet \frac{3}{51} \bullet \frac{48}{50} = \frac{576}{132600}$

Some events can take place in different ways. To calculate the total probability, the probability of each of the possible ways is added together.

❼ Calculate the probability of an event that can occur in different ways.

1. Identify each different way it can occur.
2. Calculate the probability of each way (see 4-C).
3. Add these probabilities together.

❼ Giselle grabs 3 random pens from a drawer with 6 black pens, 4 red pens, and 1 purple pen. What is the probability that they are all the same color?

1. It could be black, black, black, or it could be red, red, red.
2. $P(\text{all black}) = \frac{6}{11} \bullet \frac{5}{10} \bullet \frac{4}{9} = \frac{120}{990}$
   $P(\text{all red}) = \frac{4}{11} \bullet \frac{3}{10} \bullet \frac{2}{9} = \frac{24}{990}$
3. $P(\text{all black or all red}) = \frac{120}{990} + \frac{24}{990} = \frac{144}{990}$

When the different ways in which an event can happen are simply different orders of the same thing, each way will have the same probability. Therefore, instead of separately identifying each order and calculating its probability, the probability can be calculated once and multiplied by the number of possible orders.

❽ Calculate the probability of an event that can occur in different orders.

1. Identify one possible order.
2. Calculate the probability of this order.
3. Count or calculate the number of possible orders, using $\binom{n}{r}$ as needed.
4. Multiply the number of possible orders by the probability of each order.

❽ What is the probability that Giselle's 3 pens, above, are all different colors?

1. One possible order is black, red, purple.
2. $P(\text{black, red, purple}) = \frac{6}{11} \bullet \frac{4}{10} \bullet \frac{1}{9} = \frac{24}{990}$
3. There are three spots to choose from for the black pen, two spots left for the red pen, and one remaining for the purple pen, for a total of $\binom{3}{1}\binom{2}{1}\binom{1}{1} = 3 \bullet 2 \bullet 1 = 6$ (that is, $_3P_3$) possible orders.
4. $P(\text{all different colors}) = 6 \bullet \frac{24}{990} = \frac{144}{990}$

An event must either happen or not happen. Therefore, these two possibilities are complements of each other, making the sum of their probabilities 100%: $P(A) + P(A') = 1$.

Often it is simpler to calculate $P(A')$ than $P(A)$, especially if A can happen in more ways than A' can. In this case, $P(A)$ can be found by subtracting its complement from 1: $P(A) = 1 - P(A')$. For example, the probability of getting one or more tails in four coin flips could be calculated by simply $1 - P(0)$ rather than by $P(1) + P(2) + P(3) + P(4)$.

❾ Calculate the probability of at least or at most x out of n occurrences of an event.

1. Count how many ways the event can occur and how many ways its complement can occur to identify whether it will be simpler to calculate the probability of the outcome as stated or the probability of its complement.
2. Calculate each probability identified in step 1 (see 4-C).
3. Calculate the sum of these probabilities.
3. If you calculated the probability of the complement, subtract it from 1.

❾ Find the probability that out of five 6-sided dice, at least two will roll a 6.

1. The outcome as stated involves the sum of four different probabilities: $P(2) + P(3) + P(4) + P(5)$. The complement is only two: $P(0) + P(1)$.
2. $P(0) = \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} = \frac{3125}{7776}$
   $P(1) = \binom{5}{1} \frac{1}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} = \frac{3125}{7776}$
3. $P(0) + P(1) = \frac{3125}{7776} + \frac{3125}{7776} = \frac{6250}{7776}$
4. $P(\geq 2) = \frac{7776}{7776} - \frac{6250}{7776} = \frac{1526}{7776}$

4-C Expected Value

The EXPECTED VALUE of an event is $\mu = \Sigma xP(x)$, which is the theoretical average result.

When betting against a casino ("the house"), the bettor's expected value is the average amount expected to be gained overall on each bet. It can be found by subtracting the price of the game from the value calculated above. This is the same as the average amount expected to be lost overall by the house on each bet. It is commonly negative, meaning the bettor is expected to lose money and the house is expected to gain money.

❶ Calculate the expected value of a casino event.

1. Identify the probability of each outcome.
2. Multiply the probability of each outcome by its value.
3. Add these products together.
4. Subtract the cost to play.

❶ Connor pays \$5 to draw a card. He wins \$10 if it is a king, and he wins \$100 if it is a black ace.

Event	Probability	Value	Product	
king	$\frac{4}{52}$	\$10	\$0.77	
black ace	$\frac{2}{52}$	\$100	\$3.85	
other		\$0	\$0.00	
			\$4.62	He wins an average of \$4.62 each game.
			– \$5.00	He pays \$5.00 to play each game.
			-\$0.38	Overall, he loses an average of \$0.38 each game.

Note that \$0.38 can be written as 38¢ (38 cents) but not as .38¢ (less than one cent).