Chapter Eight

Probability

Probability of a Single Event

Probability of Multiple Events

Combinations

The Probability of *A*

The probability of an event *A* is the number of possible successful outcomes divided by the total number of equally likely possible outcomes. The set of all possible outcomes is called the **sample space**.

Scenario	Event	Sample Space	Probability
Choose a random day of the week.	The day is Thursday.	Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday	$\frac{1}{7}$
Roll a 6-sided die.	The roll is higher than 2.	1, 2, 3, 4, 5, 6	$\frac{4}{6}$
Roll two 4-sided dice.	The total is 6 or higher.	1+1, 1+2, 1+3, 1+4, 2+1, 2+2, 2+3, 2+4, 3+1, 3+2, 3+3, 3+4, 4+1, 4+2, 4+3, 4+4	$\frac{6}{16}$

It is not incorrect to reduce probabilities or convert them to decimals or percents. However, doing so removes information about the event, and it is not permitted during this chapter except for fractions equaling 0 or 1.

The Hardest Easy Problem

If there are n items and r of them are selected at random, each of the n items has a probability of $\frac{r}{n}$ of being one of the r selected items.

Sample space	Problem	Probability
Seven days of the week	Choose two random days. What is the probability that Thursday is one of the two chosen days?	$\frac{2}{7}$
Six white marbles and one red marble	Choose two random marbles. What is the probability that the red marble gets chosen?	$\frac{2}{7}$
A drawing with seven entrants, including Brody	There are two winners. What is the probability that Brody is one of the winners?	$\frac{2}{7}$

This type of problem tends to be one of the most difficult for people to understand.

The Complement of an Event

The COMPLEMENT of event A is event "not A", written A'. A' includes all possible outcomes that are not included in A, so A and A' combined include all possible outcomes: $P(A) + P(A') = 100\%$.

The "opposite" of A is not necessarily its complement. It is only the complement if there are no possible outcomes other than A and A'.

It is often easier to calculate $P(A')$ than $P(A)$. In this case, $P(A)$ can be calculated as $\mathbf{P(A) = 1 - P(A')}$.

Event A	**Complement A'**	**Incorrect Complement**
It is raining.	It is not raining.	It is sunny.
All of the cards are clubs.	At least one of the cards is not clubs (that is, not all of the cards are clubs).	None of the cards are clubs.
None of the cards are clubs.	At least one of the cards is clubs (that is, not all of the cards are not clubs).	All of the cards are clubs.

The Probability of *A* or *B*

In general, the word **or** in probability indicates addition.

The probabilities of events that cannot both occur simultaneously can be added to find the probability that one of the events will occur: $P(A \text{ or } B) = P(A) + P(B)$. Such events are **mutually exclusive**.

However, using this approach for two events that can occur simultaneously results in double-counting the amount of the overlap. This can be fixed by subtracting the amount by which the two events overlap: $\boldsymbol{P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)}$.

The examples below refer to a card drawn from a standard 52-card deck.

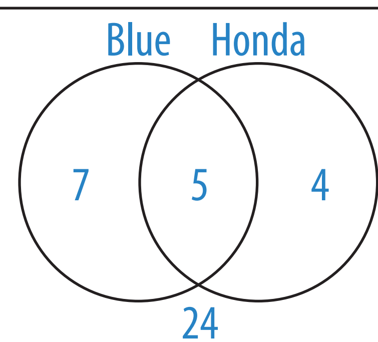
Events	Probabilities	Mutually Exclusive	Probability of *A* or *B*
***A*: king** ***B*: queen**	$P(A) = \frac{4}{52}$ $P(B) = \frac{4}{52}$	Yes: A card could be a king or a queen, but not both.	$P(A \text{ or } B) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52}$
***A*: king** ***B*: hearts**	$P(A) = \frac{4}{52}$ $P(B) = \frac{13}{52}$	No: There is one card that is both a hearts and a king.	$P(A \text{ or } B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$
***A*: king** ***B*: red**	$P(A) = \frac{4}{52}$ $P(B) = \frac{26}{52}$	No: There are two red kings.	$P(A \text{ or } B) = \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52}$

Two-Way Tables and Venn Diagrams

A **two-way table** shows the relationship between two variables. The conditions of one variable are listed as rows, the conditions of the other variable are listed as columns, and there is a row and a column for the totals. Missing values can be found by adding or subtracting.

If both variables in a two-way table have only two conditions, the values (without the totals) can be shown in a **Venn diagram**, which shows a circle for each variable.

In the examples below, a parking garage has 40 cars. 12 of the cars are blue, 9 are Hondas, and 24 are neither blue nor a Honda. What is the probability that a random car is blue or a Honda?

<table>
<tr><th>Visual Aid</th><th>Sketch</th><th>*P*(blue or Honda)</th></tr>
<tr><td>Two-Way Table</td><td>
<table>
<tr><th></th><th>Blue</th><th>Not Blue</th><th>Total</th></tr>
<tr><td>Honda</td><td>5</td><td>4</td><td>9</td></tr>
<tr><td>Not Honda</td><td>7</td><td>24</td><td>31</td></tr>
<tr><td>Total</td><td>12</td><td>28</td><td>40</td></tr>
</table>
</td><td>There are 9 Hondas plus 7 other blue cars.
$\frac{9}{40} + \frac{7}{40} = \frac{16}{40}$</td></tr>
<tr><td>Venn Diagram</td><td>
</td><td>All the cars in the circles are blue or Hondas (or both).
$\frac{7}{40} + \frac{5}{40} + \frac{4}{40} = \frac{16}{40}$</td></tr>
</table>

The Probability of *A* and *B*

In general, the word **and** in probability indicates multiplication.

The probabilities of events that are unrelated can be multiplied to find the probability that both events will occur: $P(A \text{ and } B) = P(A) \times P(B)$. Unrelated events are **independent**.

In some cases, the probability of an event changes based on knowledge of other events. Such events are **dependent**.

Events	Probabilities	Independence	Probability of *A* and *B*
A: 1st flip is heads **B: 2nd flip is heads**	$P(A) = \frac{1}{2}$ $P(B) = \frac{1}{2}$	Independent: The two coin flips have nothing to do with each other.	$P(A \text{ and } B) =$ $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
A: 1st card is a king **B: 2nd card is a king**	$P(A) = \frac{4}{52}$ $P(B) = \frac{3}{51}$	Dependent: There are only 3 kings left after one is identified.	$P(A \text{ and } B) =$ $\frac{4}{52} \times \frac{3}{51} = \frac{12}{2652}$
A: 1st card is a king **B: 5th card is a king**	$P(A) = \frac{4}{52}$ $P(B) = \frac{3}{51}$	Dependent: There are only 3 kings left after one is identified. (The 2nd, 3rd, and 4th cards are unknown, so they don't affect anything.)	$P(A \text{ and } B) =$ $\frac{4}{52} \times \frac{3}{51} = \frac{12}{2652}$

Outcomes that can occur in different ways

Some probability problems involve both "and" and "or", even if those words are not used in the problem.

In the examples below, two candies are taken from a bag of 6 orange, 4 cherry, and 3 grape candies.

Event	How it can occur	Probability
The candies are the same flavor.	The first one is orange and the second one is orange or the first one is cherry and the second one is cherry or the first one is grape and the second one is grape.	$\frac{6}{13} \times \frac{5}{12} + \frac{4}{13} \times \frac{3}{12} + \frac{3}{13} \times \frac{2}{12}$ $= \frac{30}{156} + \frac{12}{156} + \frac{6}{156}$ $= \frac{48}{156}$
The candies are not the same flavor.	The first one is orange and the second one is not orange or the first one is cherry and the second one is not cherry or the first one is grape and the second one is not grape.	$\frac{6}{13} \times \frac{7}{12} + \frac{4}{13} \times \frac{9}{12} + \frac{3}{13} \times \frac{10}{12}$ $= \frac{42}{156} + \frac{36}{156} + \frac{30}{156}$ $= \frac{108}{156}$

For scenarios involving a small number of events, a tree diagram can aid in visualizing the possible outcomes.

Factorials

***x*!** is the number ***x* factorial**.

For positive integer values of x, $x!$ is the product of every integer from 1 to x: $x! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot x$.

For other values of x, $x!$ is undefined, except $0! = 1$.

When dividing factorials, many factors will cancel and do not need to be written.

Example	Factors	Result
$6!$	$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$	720
$\frac{6!}{4!}$	$\frac{6 \cdot 5 \cancel{\cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{4 \cdot 3 \cdot 2 \cdot 1}}$	30
$\frac{6!}{4!\,2!}$	$\frac{6 \cdot 5 \cancel{\cdot 4 \cdot 3 \cdot 2 \cdot 1}}{1 \cdot 2 \cancel{\cdot 4 \cdot 3 \cdot 2 \cdot 1}}$	15
$0!$	1	1

Combinations

A **combination** is a group of selected items.

The number of possible combinations of r items in a group of n items is ***n* choose *r***, written $\binom{n}{r}$ or ${}_nC_r$.

The formula to count combinations is $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$.

Since many of the factors will cancel when using this formula, an easier way to think about it is to put r factors in the numerator, starting at n and counting down, and r factors in the denominator, starting at 1 and counting up. $(n - r)$ can be used in place of r.

Group of n items	Chosen group of r items	Combination Example	Number of possible combinations
7 days of the week	1 day	Friday	$\binom{7}{1} = \frac{7}{1} = 7$
7 days of the week	2 days	Friday and Sunday	$\binom{7}{2} = \frac{7 \cdot 6}{1 \cdot 2} = 21$
7 days of the week	5 days	every day except Friday and Sunday	$\binom{7}{5} = \binom{7}{2} = \frac{7 \cdot 6}{1 \cdot 2} = 21$
12 months of the year	4 months	June, September, October, December	$\binom{12}{4} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} = 495$

The Fundamental Counting Principle

The **fundamental counting principle** states that if two independent events have *a* and *b* possible outcomes, respectively, then there are a total of ***ab*** possible outcomes for the two events. This can be expanded to *abc* possible outcomes for three events, etc.

For example, there are $6 \times 6 = 36$ possible outcomes when rolling two 6-sided dice (1-1, 1-2, 1-3, etc.).

In the examples below, 5 juniors and 7 seniors are competing in an art contest.

Chosen winners	What is being chosen	# of possible outcomes
2 juniors, **2 seniors**	2 of the 5 juniors 2 of the 7 seniors	$\binom{5}{2}\binom{7}{2} =$ $\left(\frac{5 \cdot 4}{1 \cdot 2}\right)\left(\frac{7 \cdot 6}{1 \cdot 2}\right) =$ $10 \cdot 21 = 210$
1st place, **2nd place,** **2 runners up**	1 of the 12 people for 1st place 1 of the remaining 11 people for 2nd place 2 of the remaining 10 people for runner up	$\binom{12}{1}\binom{11}{1}\binom{10}{2} =$ $\left(\frac{12}{1}\right)\left(\frac{11}{1}\right)\left(\frac{10 \cdot 9}{1 \cdot 2}\right) =$ $12 \cdot 11 \cdot 45 = 5940$
1st place, **2nd place,** **3rd place**	1 of the 12 people for 1st place 1 of the remaining 11 people for 2nd place 1 of the remaining 10 people for 3rd place	$\binom{12}{1}\binom{11}{1}\binom{10}{1} =$ $\left(\frac{12}{1}\right)\left(\frac{11}{1}\right)\left(\frac{10}{1}\right) =$ $12 \cdot 11 \cdot 10 = 1320$

A series of individually identified items is called a **permutation**. The formula $_nP_r = \frac{n!}{(n-r)!}$ counts permutations by multiplying *r* integers together, starting with *n* and going down, such as $12 \cdot 11 \cdot 10$ in the last example.