CHAPTER EIGHT: PROBABILITY

Test: Tuesday, May 12

The probability of an event is the number of ways it can successfully happen divided by the total possible number of equally likely ways. Many people find probability to be difficult because it involves reasoning and can be counterintuitive. For example, people commonly do not understand that probability is based on what information is known, regardless of what events have actually occurred. On the other hand, its direct real-world applicability can make it more appealing than algebra-based topics.

8-A Probability of a Single Event — Tuesday • 4/14

sample space • complement • mutually exclusive • two-way table • Venn diagram

❶ List the sample space of an event.
❷ Use a sample space to find the probability of an event.
❸ Find the probability that a specific item is one of the items selected from a group.
❹ Use a complement to find the probability of an event.
❺ Find the probability that either of two specific events will occur.
❻ Find probabilities based on given information.
❼ Use a two-way table to organize data and determine basic probabilities involving two variables.
❽ Use a two-way table to organize data and determine conditional probabilities involving two variables.
❾ Create a Venn diagram representing two variables with two conditions each.

8-B Probability of Multiple Events — Tuesday • 4/21

dependent events • independent events

❶ Find the probability of multiple events all occurring.
❷ Identify whether events are independent or dependent.
❸ Find the probability of multiple dependent events all occurring.
❹ Calculate the probability of an event that can happen in different ways.

8-C Combinations — Tuesday • 4/28

combination • choose • fundamental counting principle • permutation

❶ Count combinations.
❷ Find the total number of possible outcomes in a series of events.
❸ Count permutations.

8-A Probability of a Single Event

A SAMPLE SPACE shows all possible outcomes of an event.

❶ List the sample space of an event.

1. List all possible outcomes. If the event consists of more than one action, make sure to include every possible combination.

❶ a) Roll a 6-sided die. b) Flip three coins

1, 2, 3, 4, 5, 6 HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

The probability of an event A is written $P(A)$ and is a number between 0 and 1 (that is, between 0% and 100%).

If each of the n possible outcomes are equally likely and r of the n possible outcomes are considered successes, then the probability of success is $P(A) = \frac{r}{n}$.

Reducing a probability, or converting it to a decimal or percentage, removes information from the answer because r and n are no longer known. Therefore, during this chapter it is required to express all probabilities as unreduced fractions unless directed otherwise, except for those equaling 0 or 1.

❷ Use a sample space to find the probability of an event.

1. Count the number n of equally likely possible outcomes. See 8-C if needed.
2. Count the number r of these n outcomes that are part of the desired event.
3. The probability of event A occurring is $P(A) = \frac{r}{n}$.

❷ Find the probability of rolling higher than 17 on the dice stated.

a) a single 20-sided die

1. There are $n = 20$ equally likely possible outcomes.
2. Of these 20 outcomes, $r = 3$ of them (18, 19, and 20) are higher than 17.
3. $P(x > 17) = \frac{3}{20}$

b) two 10-sided dice added together

1. There are $n = 10 \times 10 = 100$ equally likely possible outcomes (see 8-C).
2. Of these 100 outcomes $r = 6$ of them (10+8, 9+9, 8+10, 10+9, 9+10, and 10+10) are higher than 17.
3. $P(x > 17) = \frac{6}{100}$

A special case of this type of problem is where there are n possible outcomes, r of which take place. The probability that a specific one of those n possible events is one of the r events that actually takes place is r out of n. For example, if six different letters of the alphabet are randomly selected, there are 6 chances out of 26 that D is one of the letters selected (and 20 chances out of 26 that it is not).

❸ Find the probability that a specific item is one of the items selected from a group.

1. Identify the number of possibilities n.
2. Identify the number of selected elements r.
3. $P(r) = \frac{r}{n}$

❸ If three random US states are selected, what is the probability that one of them is Oregon?

1. There are $n = 50$ states.
2. $r = 3$ states are selected.
3. $P(\text{Oregon}) = \frac{3}{50}$

The COMPLEMENT of an Event is all possible outcomes in the sample space other than the event itself. The complement of event A is written A', meaning "not A".

A' includes all possible outcomes in the sample space not in A. Therefore, $P(A) + P(A') = 100\%$.

$P(A)$ and $P(A')$ can both be calculated by subtracting the other from 1: $P(A) = 1 - P(A')$, and vice versa.

❹ Use a complement to find the probability of an event.

1. Identify the complement. It can be found by adding "not" appropriately to the original event. It is only the complement if there are no possible outcomes other than it and the original event.
2. Find the probability of the complement, $P(A')$.
3. Subtract this probability from 1.

❹

a) Out of three coin flips, at least one is heads.

1. The complement is *None of the three coin flips are heads.*
2. $P(\text{no heads}) = \frac{1}{8}$ (see ❶ b).
3. $P(\text{at least one heads}) = \frac{8}{8} - \frac{1}{8} = \frac{7}{8}$

b) If three random US states are selected, what is the probability that none of them is Oregon?

1. The complement is *One of the three states is Oregon.*
2. $P(\text{Oregon}) = \frac{3}{50}$ (see ❸)
3. $\frac{50}{50} - \frac{3}{50} = \frac{47}{50}$

MUTUALLY EXCLUSIVE (or Disjoint) events cannot happen simultaneously.

If events A and B are mutually exclusive, such as rolling a 2 and rolling an odd number, then $P(A \text{ or } B) = P(A) + P(B)$. However, if events A and B could happen simultaneously, such as rolling a a number below 5 and rolling an odd number, this approach will not work because some outcomes are double-counted.

❺ Find the probability that either of two specific events will occur.

1. Identify the probability of each event. Label them $P(A)$ and $P(B)$.
2. Identify the probability of the two events happening simultaneously. Label it $P(A \text{ and } B)$.
3. Calculate $P(A) + P(B) - P(A \text{ and } B)$.

❺ A card is drawn from a deck. Calculate the probability of it being as stated.

a) hearts or black

1. $P(\text{hearts}) = \frac{13}{52}$, $P(\text{black}) = \frac{26}{52}$
2.
3. $P(\text{hearts or black}) = \frac{13}{52} + \frac{26}{52} = \frac{39}{52}$

b) hearts or a queen

$P(\text{hearts}) = \frac{13}{52}$, $P(\text{queen}) = \frac{4}{52}$

$P(\text{queen of hearts}) = \frac{1}{52}$

$P(\text{hearts or queen}) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52}$

A CONDITIONAL Probability is one in which a given condition reduces the size of the sample space. The word GIVEN is a mathematical term meaning *known* or *knowing*.

❻ Find probabilities based on given information.

1. Do not assume any information, even for events that have already occurred.
2. Take into account all known (given) information, even for events that have not yet occurred.

❻ Find the following probabilities for two randomly drawn cards.

a) The second card is an ace.

1. The first card remains unknown. Do no assume it is or is not an ace. There are still $n = 52$ possible cards, $r = 4$ of which are aces, so $P(A) = \frac{4}{52}$.

b) The second card is an ace, given the first card is an ace.

2. The first card is known. There are only $n = 51$ remaining possiblities for the second card, $r = 3$ of which are aces, so $P(A) = \frac{3}{51}$.

c) The first card is an ace, given the second card will be an ace.

2. The second card is known even though it hasn't been flipped yet. There are only $n = 51$ remaining possiblities for the first card, $r = 3$ of which are aces, so $P(A) = \frac{3}{51}$.

A TWO-WAY Table shows the frequency of each possible outcome, including overlapping outcomes, for situations with two variables.

❼ Use a two-way table to organize data and determine basic probabilities involving two variables.

1. Make a column for each condition of one variable, and make a row for each condition of the other variable.
2. Add in a column for the row totals and a row for the column totals.
3. Fill in the given information.
4. Add or subtract known values to find other values that are unknown.
5. To find a probability, add the appropriate cells in the table to determine *r* for the numerator. The denominator *n* is the total.

❼ In a neighborhood of 70 people, 20 out of 32 of the males and 25 out of 38 of the females are children. Use this information to fill in a two-way table and determine the probability that a random person from the neighborhood meets the stated criteria.

a) a child b) a female c) a child or a female d) a child or a male

1. For the variable of sex, there is a Male column and a Female column. For the variable of age, there is a Child row and an Adult row.
3. We know there are a total of 32 males, 20 of whom are children, and we know there are a total of 38 females, 25 of whom are children.
4. Since 20 of the 32 males are children, the other $32 - 20 = 12$ males are adults. Likewise, the remaining $38 - 25 = 13$ females are adults. There are a total of $20 + 25 = 45$ children, $12 + 13 = 25$ adults, and $32 + 38 = 70$ people altogether.
5. a) 45 of the 70 people are children: $P(\text{child}) = \frac{45}{70}$
   b) 38 of the 70 people are female: $P(\text{female}) = \frac{38}{70}$
   c) $20 + 25 + 13 = 58$ of the people are children or female: $P(\text{child or female}) = \frac{58}{70}$
   d) $20 + 25 + 12 = 57$ of the people are children or male: $P(\text{child or male}) = \frac{57}{70}$

	Female	Male	**Total**
Child	25	20	**45**
Adult	13	12	**25**
Total	**38**	**32**	**70**

Two-way tables are often helpful for conditional probabilities.

❽ Use a two-way table to organize data and determine conditional probabilities involving two variables.

1. Follow the steps for two-way tables, above.
2. Identify the denominator based on the given information. It will be the row total or the column total instead of the overall total.

❽ Determine the probability of the following for the scenario above.

a) A random *child* is female. b) A random *female* is a child. c) A random *person* is a female child.

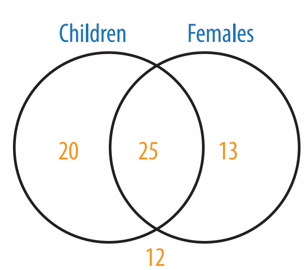
2. The denominator is the 45 children. The denominator is the 38 females. The denominator is the 70 people.
   $P(\text{female, given child}) = \frac{25}{45}$ $P(\text{child, given female}) = \frac{25}{38}$ $P(\text{female and child}) = \frac{25}{70}$

When both variables in a two-way table each have only two possible values, the data can also be shown in a VENN DIAGRAM, which has one circle for each variable. The values in a Venn diagram can be determined by making a two-way table.

❾ Create a Venn diagram representing two variables with two conditions each.

1. Make a two-way table (see ❽).
2. Draw two overlapping circles, with one of them labeled as one of the row labels and the other labeled as one of the column labels in the two-way table.
3. Put each of the four values in the two-way table (other than the totals) in its corresponding place in the Venn diagram. One of the values will be outside the circles.

❾ Put the information from ❽ in a Venn diagram.



8-B Probability of Multiple Events

The probability of independent events A and B both occurring is $P(A \text{ and } B) = P(A) \times P(B)$.

❶ Find the probability of multiple independent events all occurring.

1. Identify the probability of each event being asked about.
2. Multiply these probabilities together.

❶ Gabby rolls three 6-sided dice. Calculate the following probabilities.

a) The first roll is a 2, the second roll is a 5, and the third roll is odd.

1. $P(2) = \frac{1}{6}$, $P(5) = \frac{1}{6}$, $P(\text{odd}) = \frac{3}{6}$
2. $P(2, 5, \text{odd}) = \frac{1}{6} \times \frac{1}{6} \times \frac{3}{6} = \frac{3}{216}$

b) All three rolls are lower than 3.

1. $P(<3) = \frac{2}{6}$
2. $P(<3, <3, <3) = \frac{2}{6} \times \frac{2}{6} \times \frac{2}{6} = \frac{8}{216}$

c) The first roll is a 6 and the third roll is a 4.

1. $P(6) = \frac{1}{6}$, $P(4) = \frac{1}{6}$
2. $P(6, 4) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ (Note that the second roll is not being asked about so it is not part of the calculations.)

The probabilities of DEPENDENT Events change based on what is known about other events.

The probabilities of INDEPENDENT Events do not change.

❷ Identify whether events are independent or dependent.

1. If the probability of events are the same on every trial, the events are independent.
2. If the outcome of an event changes the sample space for other events, the events are dependent.

❷ Explain whether the following are independent or dependent.

a) drawing two cards

2. Dependent: Once one card is known, there are only 51 cards remaining in the sample space that the other card could be.

b) flipping two coins

1. Independent: Each coinflip has a 50% chance of heads and a 50% chance of tails, regardless of what the other coinflip was.

The probability of dependent events A and B both occurring is $P(A \text{ and } B) = P(A) \times P(B \text{ given } A)$, where $P(B \text{ given } A)$ is the conditional probability of B taking into account that A is true.

❸ Find the probability of multiple dependent events all occurring.

1. Identify the probability of the first event being asked about.
2. Identify the probability of the second event being asked about, taking into account that the first event has occured. In basic cases, this means subtracting one from the denominator and usually subtracting one from the numerator as well.
3. Continue identifying each probability, taking into account all previous events.
4. Multiply these probabilities together.

❸ Gabby draws three cards. Calculate the probability that all three are aces.

1. $P(\text{an ace}) = \frac{4}{52}$
2. $P(\text{a second ace}) = \frac{3}{51}$
3. $P(\text{a third ace}) = \frac{2}{50}$
4. $P(\text{three aces}) = \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} = \frac{24}{132600}$

Some events can occur in different ways. The probability of such an event is the sum of the probabilities of the different ways it can happen.

❹ Calculate the probability of an event that can happen in different ways.

1. Identify the different ways in which the event could occur.
2. Calculate the probability of each of these different ways.
3. Add these probabilities together.

❹ A class has 9 freshmen, 20 sophomores, and 7 juniors. What is the probability that two random students are in the same grade?

1. They could be two freshmen, two sophomores, or two juniors.
2. $P(F, F) = \frac{9}{36} \times \frac{8}{35} = \frac{72}{1260}$
   $P(S, S) = \frac{20}{36} \times \frac{19}{35} = \frac{380}{1260}$
   $P(J, J) = \frac{7}{36} \times \frac{6}{35} = \frac{42}{1260}$
3. $P(\text{same grade}) = \frac{72}{1260} + \frac{380}{1260} + \frac{42}{1260} = \frac{494}{1260}$

8-C Combinations

The FACTORIAL of a natural number x is the product of the integers from 1 to x, and is written $x!$. $x! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot x$

Factorials of negative numbers and nonintegers are not defined, but $0! = 1$.

A COMBINATION is a group of items not assigned specific labels or placements within the group. N CHOOSE R, written $\binom{n}{r}$ or ${}_nC_r$, is the number of combinations of r items that can be made from a set of n items. $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, which can be calculated using nCr on a calculator. Note that $\binom{n}{0} = 1$, $\binom{n}{n} = 1$, and $\binom{n}{r} = \binom{n}{n-r}$.

❶ Count combinations.

1. Identify n, the number of items there are to choose from.
2. Identify r, the number of items being chosen. Since $\binom{n}{r}$ and $\binom{n}{n-r}$ are the same thing, you can use $n-r$ instead of r when $n-r$ is smaller than r.
3. Multiply r integers together in the denominator, starting with 1 and going up.
4. Multiply r integers together in the numerator, starting with n and going down.
5. Simplify.

❶ In how many ways can Georgia choose her 6 favorite songs from a playlist of 9 songs?

1. $n = 9$
2. $r = 6$, or use $9 - 6 = 3$ instead.

3, 4, 5. $\binom{9}{6} = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = \frac{504}{6} = 84$

The FUNDAMENTAL COUNTING PRINCIPLE states that if event A has a possible outcomes and event B has b possible outcomes, there are a total of ab possible outcomes in the sample space.

❷ Find the total number of possible outcomes in a series of events.

1. Identify the size of the sample space of each individual event, using $\binom{n}{r}$ as needed.
2. Multiply these sizes together.

❷ State the number of possible outcomes of the following.

a) Choose 3 representatives out of 9 seniors and 2 representatives out of 8 juniors.

$\binom{9}{3}\binom{8}{2} = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} \cdot \frac{8 \cdot 7}{1 \cdot 2} = 84 \cdot 28 = 2352$

b) Identify the 1st place, 2nd place, 3rd place, and 4th place finisher out of 25 racers.

$\binom{25}{1}\binom{24}{1}\binom{23}{1}\binom{22}{1} = \frac{25}{1} \cdot \frac{24}{1} \cdot \frac{23}{1} \cdot \frac{22}{1} = 25 \cdot 24 \cdot 23 \cdot 22 = 303{,}600$

The outcome of a series of events that each involve choosing one element from those remaining, as in example ❷b above, is called a PERMUTATION. This is the same as a combination, except that each item in the group is assigned a specific label or placement within the group. In other words, a combination is a selection of r items from a group of n items, and a permutation is a selection of r uniquely identified items from a group of n items. The unique identifiers are sometimes order of the items, such as first, second, third, and fourth in the above example.

Permutations are never needed, but they can be used as an easier way to indicate multiple combinations in which one of the remaining elements is chosen each time. For example, $\binom{25}{1}\binom{24}{1}\binom{23}{1}\binom{22}{1}$ can be written as ${}_{25}P_4$.

The number of permutations of r items that can be made from a set of n items is ${}_nP_r = \frac{n!}{(n-r)!}$, which can be calculated using nPr on a calculator.

❸ Count permutations.

1. Identify n, the number of items there are to choose from.
2. Identify r, the number of items being chosen.
3. Multiply r integers together, starting with n and going down.

❸ In how many ways can Georgia choose her favorite, second favorite, and third favorite song from a playlist of 9 songs?

1. $n = 9$
2. $r = 3$
3. ${}_9P_3 = 9 \cdot 8 \cdot 7 = 504$