Paper 1 Mock Mock [110 marks]

1. [Maximum mark: 5] 25M.1.AHL.TZ1.1

Consider the function $f(x) = \frac{4x^3}{3} - 16x$, where $x \in \mathbb{R}$.

The graph of $y = f(x)$ has a local minimum point at $(p,\ q)$ where $p > 0$.

Find the value of p and the value of q. [5]

Markscheme

$f\prime(x) = 4x^2 - 16$ ***A1***

sets their derivative equal to zero ***(M1)***

$4x^2 - 16 = 0, (x = \pm 2)$

$p = 2$ (accept $x = 2$) ***A1***

substitutes their **positive** p into $f(x)$ ***(M1)***

$y = \frac{4(2^3)}{3} - 16(2) \left(= \frac{32}{3} - 32 = -\frac{64}{3}\right)$

$q = -\frac{64}{3}$ (accept $y = \frac{-64}{3}$) ***A1***

[5 marks]

2. [Maximum mark: 7] 25M.1.AHL.TZ1.2

Bob invests $1\,000$ dinar in an account which pays a nominal annual interest rate of $4\,\%$ compounded **quarterly**.

The amount of money in the account after one complete year can be written as $1\,000(1 + k)^4$ where $k \in \mathbb{Q}$.

(a) Write down the value of k. [1]

Markscheme

$k = \frac{4}{400} \left(= \frac{1}{100} = 0.01\right)$ ***A1***

[1 mark]

(b) Expand and simplify $(1 + x)^4$. [2]

Markscheme

attempt to find binomial coefficients or multiply out brackets ***(M1)***

e.g. Pascal's triangle down to correct row **OR** $\left(1 + 2x + x^2\right)^2$ **OR** substitute into binomial expansion

$(1 + x)^4$

$= 1 + 4x + 6x^2 + 4x^3 + x^4$ ***A1***

[2 marks]

(c) Hence or otherwise, find the amount of money in the account after one complete year, giving your answer correct to the nearest dinar. [4]

Markscheme

METHOD 1

recognition that the expansion can be used with x replaced with k ***(M1)***

$\left(1 + \frac{1}{100}\right)^4$

$= 1 + \frac{4}{100} + \frac{6}{100^2} + \ldots (= 1 + 0.04 + 0.0006 + \ldots)$ ***(A1)***

multiplies by 1000 (seen anywhere) ***(M1)***

$1000\left(1 + \frac{1}{100}\right)^4$

$1000 + 40 + 0.6 + \ldots (= 1040.6\ldots)$

$= 1041$ (dinar) ***A1***

METHOD 2

attempt to find the value of $(1 + k)^4$ by hand ***(M1)***

$(1.01)^4 = (1.0201)(1.01)^2 = (1.030301)(1.01)$

$= 1.0406\ldots$ ***(A1)***

multiplies by 1000 (seen anywhere) ***(M1)***

$1000(1.01)^4$

$= 1040.6\ldots$

$= 1041$ (dinar) ***A1***

[4 marks]

3. [Maximum mark: 4] 25M.1.AHL.TZ1.3

Find the area completely enclosed by the curves $y = \mathrm{e}^x$, $y = -\mathrm{e}^x$, and the lines $x = -1$ and $x = 1$. [4]

Markscheme

METHOD 1

attempt to set up integral $\mathrm{e}^x - (-\mathrm{e}^x) = 2\mathrm{e}^x$ or $2\mathrm{e}^x$ and then double ***(M1)***

$\int(\mathrm{e}^x - (-\mathrm{e}^x))\ \mathrm{d}\ x$ **OR** $2\int \mathrm{e}^x\ \mathrm{d}\ x$

$= 2\int_{-1}^{1} \mathrm{e}^x\ \mathrm{d}\ x$

$= 2[\mathrm{e}^x]_{-1}^{1}$ ***(A1)***

attempt to substitute correct limits into their integrated function and subtract ***(M1)***

$= 2\left(\mathrm{e} - \frac{1}{\mathrm{e}}\right), 2\mathrm{e} - \frac{2}{\mathrm{e}}, 2\mathrm{e} - 2\mathrm{e}^{-1}$ ***A1***

METHOD 2

$\int_{-1}^{1} \mathrm{e}^x\ \mathrm{d}\ x = [\mathrm{e}^x]_{-1}^{1}$ and $\int_{-1}^{1} -\mathrm{e}^x\ \mathrm{d}\ x = [-\mathrm{e}^x]_{-1}^{1}$ ***(A1)***

attempt to substitute correct limits into both their integrated functions and subtract ***(M1)***

$\mathrm{e}^1 - \mathrm{e}^{-1}$ and $-\mathrm{e}^1 - \left(-\mathrm{e}^{-1}\right)$

subtracts their two integrals in correct order ***(M1)***

$\mathrm{e}^1 - \mathrm{e}^{-1} - \left(-\mathrm{e}^1 + \mathrm{e}^{-1}\right)$

$= 2\left(\mathrm{e} - \frac{1}{\mathrm{e}}\right), 2\mathrm{e} - \frac{2}{\mathrm{e}}, 2\mathrm{e} - 2\mathrm{e}^{-1}$ ***A1***

[4 marks]

4. [Maximum mark: 6] 25M.1.AHL.TZ1.4

Consider events A and B such that $\mathrm{P}(A') = \mathrm{P}(A \cup B) = \frac{3}{4}$ and $\mathrm{P}(B|A) = \frac{2}{3}$.

(a) Find $\mathrm{P}\,(A \cap B)$. [3]

Markscheme

$\mathrm{P}\,(A) = \frac{1}{4}$ ***(A1)***

attempt to use $\mathrm{P}\,(B|A) = \frac{\mathrm{P}\,(B\cap A)}{\mathrm{P}\,(A)}$ ***(M1)***

$\frac{2}{3} = \frac{\mathrm{P}\,(B\cap A)}{\left(\frac{1}{4}\right)}$

$\mathrm{P}\,(A \cap B) = \frac{2}{3}\left(\frac{1}{4}\right)$

$= \frac{2}{12}\left(= \frac{1}{6}\right)$ ***A1***

[3 marks]

(b) Show that events A and B are independent. [3]

Markscheme

attempt to use $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ **OR** a Venn diagram, with their values of $\mathrm{P}(A)$ and $\mathrm{P}(B \cap A)$ ***M1***

$\frac{3}{4} = \frac{1}{4} + \mathrm{P}(B) - \frac{1}{6}$

$\mathrm{P}(B) = \frac{1}{2} + \frac{1}{6}$

$= \frac{4}{6} \left(= \frac{2}{3}\right)$ ***A1***

$\mathrm{P}(B \mid A) = \mathrm{P}(B)$ **OR** $\mathrm{P}(A)\mathrm{P}(B) = \frac{1}{6}$ so $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$ (hence A and B are independent)
R1

Note: The ***R1*** is dependent on all previous marks.

[3 marks]

5. [Maximum mark: 8] 25M.1.AHL.TZ1.5

Consider a sequence of ten rectangular picture frames $F_1, F_2, \ldots, F_9, F_{10}$.

Picture frame F_1 has width $4\ \mathrm{cm}$ and height $5\ \mathrm{cm}$.

The width and height of picture frame F_n, are each increased by 50% to generate the width and height of the next picture frame F_{n+1}, for $n \in \mathbb{Z}^+, 1 \le n \le 9$.

(a.i) Show that the area of picture frame F_n is $20\left(\frac{9}{4}\right)^{n-1}\ \mathrm{cm}^2$. [2]

Markscheme

for a sequence of areas, uses two consecutive terms to find a common ratio **OR** for sequences of both widths and heights uses two consecutive terms for both sequences to find both common ratios **OR** recognises that both widths and heights are geometric sequences with common ratio $\frac{3}{2}$ ***M1***

areas form a geometric sequence with first term 20 and common ratio $\frac{45}{20}$ ***A1***

OR area of picture frame F_n is $4\left(\frac{3}{2}\right)^{n-1} \times 5\left(\frac{3}{2}\right)^{n-1}$

area of F_n is $20\left(\frac{9}{4}\right)^{n-1}$ ***AG***

[2 marks]

(a.ii) Hence, find the mean area of the ten picture frames, giving your answer in the form $p\left(\left(\frac{9}{4}\right)^a - 1\right)\ \mathrm{cm}^2$, where $p \in \mathbb{Q}^+, a \in \mathbb{Z}^+$. [3]

Markscheme

attempt to find the sum of the areas using $S_n = \frac{u_1(r^n - 1)}{r-1}$ ***(M1)***

sum of areas $\frac{20\left(\left(\frac{9}{4}\right)^{10} - 1\right)}{\frac{9}{4} - 1} \left(= 16\left(\left(\frac{9}{4}\right)^{10} - 1\right)\right)$ ***(A1)*** mean area $= \frac{1}{10}\left(\frac{20\left(\left(\frac{9}{4}\right)^{10}-1\right)}{\frac{9}{4}-1}\right)\left(= \frac{1}{10}\left(16\left(\left(\frac{9}{4}\right)^{10}-1\right)\right)\right)$

$= \frac{16}{10}\left(\left(\frac{9}{4}\right)^{10}-1\right)\left(= \frac{8}{5}\left(\left(\frac{9}{4}\right)^{10}-1\right)\right)$ ***A1***

$p = \frac{8}{5}, a = 10$

[3 marks]

(b) Find the median area of the ten picture frames, giving your answer in the form $q\left(\frac{9}{4}\right)^4 \text{ cm}^2$, where $q \in \mathbb{Q}^+$. [3]

Markscheme

recognition that median is between 5th and 6th picture frame ***(M1)***

median area $= \frac{20\left(\frac{9}{4}\right)^4+20\left(\frac{9}{4}\right)^5}{2}$ ***(A1)***

$= \frac{20\left(\frac{9}{4}\right)^4\left(1+\frac{9}{4}\right)}{2}$

$= \frac{65}{2}\left(\frac{9}{4}\right)^4$ ***A1***

$q = \frac{65}{2}$

[3 marks]

6. [Maximum mark: 6] 25M.1.AHL.TZ1.6

The line L_1 has vector equation $\boldsymbol{r} = 4\boldsymbol{i} - \boldsymbol{k} + \lambda(a\boldsymbol{j} + \boldsymbol{k})$, where $a, \lambda \in \mathbb{R}$.

The line L_2 has vector equation $\boldsymbol{r} = \boldsymbol{i} - b\boldsymbol{k} + \mu(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$, where $b, \mu \in \mathbb{R}$.

The lines L_1 and L_2 are perpendicular and intersect at a unique point.

Find the value of a and the value of b. [6]

Markscheme

direction vectors are $a\boldsymbol{j} + \boldsymbol{k}\left(= \begin{pmatrix}0\\a\\1\end{pmatrix}\right)$ and $\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}\left(= \begin{pmatrix}1\\2\\3\end{pmatrix}\right)$ ***(A1)***

recognition that the scalar product of the direction vectors is 0 ***(M1)***

$\begin{pmatrix}0\\a\\1\end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} (= 2a + 3) = 0$

$a = -\frac{3}{2}$ ***A1***

at point of intersection $4 = 1 + \mu$, $-\frac{3}{2}\lambda = 2\mu$ and $-1 + \lambda = -b + 3\mu$ ***(A1)***

attempt to solve 3 equations in μ, λ and b, derived from the point of intersection, to find μ, λ and b. ***(M1)***

$\mu = 3, \lambda = -4$

$b = 14$ ***A1***

[6 marks]

7. [Maximum mark: 5] 25M.1.AHL.TZ1.7

Consider the complex number $z = 3^{\mathrm{i}-1}$.

(a) Write the integer 3 in the form e^a where $a \in \mathbb{R}$. [1]

Markscheme

$(3 =)\mathrm{e}^{\ln 3}$ **OR** $a = \ln 3$ ***A1***

[1 mark]

(b) Hence, giving your answers in the form $p \cos(\ln q)$ where $p, q \in \mathbb{Q}^+$, find

(b.i) $\mathrm{Re}(z)$; [2]

Markscheme

$z = \frac{1}{3}\mathrm{e}^{\mathrm{i}\ln 3}$ **OR** $(\mathrm{Re}(z)) = \mathrm{e}^{-\ln 3}\cos(\ln 3)$ ***(A1)***

$(\mathrm{Re}(z)) = \frac{1}{3}\cos(\ln 3)$ ***A1***

[2 marks]

(b.ii) $\mathrm{Re}\left(\frac{1}{z}\right)$. [2]

Markscheme

$\frac{1}{z} = 3\mathrm{e}^{-\mathrm{i}\ln 3}(= 3(\cos(-\ln 3) + \mathrm{i}\sin(-\ln 3)))$ **OR** $\left(\mathrm{Re}\left(\frac{1}{z}\right) =\right)\mathrm{e}^{\ln 3}\cos(-\ln 3)$ ***(A1)***

$\mathrm{Re}\left(\frac{1}{z}\right) = 3\cos(-\ln 3)$

$= 3\cos(\ln 3)$ **OR** $3\cos\left(\ln\frac{1}{3}\right)$ ***A1***

[2 marks]

8. [Maximum mark: 7] 25M.1.AHL.TZ1.8

Seema claims that $n > \log_2 n$ for $n \in \mathbb{Z}^+$.

(a) Show that $1 + \log_2 n \geq \log_2(n+1)$ for $n \in \mathbb{Z}^+$. [2]

Markscheme

$1 + \log_2 n = \log_2 2 + \log_2 n$

$= \log_2(2n)$ ***A1***

$2n \geq n+1$ **OR** $\log_2(2n) \geq \log_2(n+1)$ ***R1***

(for $n \in \mathbb{Z}^+$) (since $\log_2 x$ is an increasing function)

$1+\log_2 n \geq \log_2(n+1)$ (for $n \in \mathbb{Z}^+$) ***AG***

Note: Do not award ***A0R1***.

[2 marks]

(b) Use mathematical induction and the result from part (a) to prove that Seema's claim is valid. [5]

Markscheme

for $n=1$

$\log_2 1 = 0$ and $1 > 0$ **OR** $\text{LHS} = 1, \text{RHS} = \log_2 1 = 0$ **OR** $\log_2 2 > \log_2 1$ ***R1***

(so true for $n=1$)

assume true for $n=k$, ie $k > \log_2 k$ ***M1***

Note: Award ***M0*** for statements such as "let $n=k$", "assume $n=k$ is true". The assumption of truth must be clear. "Assume $\mathrm{P_k}$ true" is accepted.
The following two marks after this ***M1*** are independent of this mark and can be awarded.

hence

$1+k > 1+\log_2 k$ (using assumption) ***M1***

$\geq \log_2 (k+1)$ (using result from part a) ***A1***

hence if true for $n=k$ then true for $n=k+1$ ***R1***

and as true for $n=1$, therefore true for all $n \in \mathbb{Z}^+$.

Note: Only award the final ***R1*** if the first three marks have been awarded.

[5 marks]

9. [Maximum mark: 8] 25M.1.AHL.TZ1.9

Consider the homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-y}{x+y}$, where $x > 0$ and $y \neq -x$.

It is given that $y=0$ when $x=2$.

By using the substitution $y=vx$, show that the solution of the differential equation is $x^2-2xy-y^2=4$. [8]

Markscheme

$y=vx$, $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v$

$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = \frac{x-vx}{x+vx} \left(= \frac{1-v}{1+v}\right)$

$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = \frac{1-v}{1+v}$ ***(A1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1-v}{1+v} - v\left(= \frac{1-2v-v^2}{1+v}\right)$

attempt to separate variables and form two integrals ***(M1)***

$\int \frac{1+v}{1-2v-v^2}\ \mathrm{d}\ v = \int \frac{1}{x}\ \mathrm{d}\ x$

$\int \frac{1}{x}\ \mathrm{d}\ x = \ln x + c$ ***A1***

use of substitution or inspection to integrate $\frac{1+v}{1-2v-v^2}$ or equivalent ***(M1)***

$u = 1 - 2v - v^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}v} = -2 - 2v = -2(1+v)$

$\int \frac{1+v}{1-2v-v^2}\ \mathrm{d}\ v = -\frac{1}{2}\ln\left|1-2v-v^2\right|$ **OR** $-\frac{1}{2}\ln\left|v^2+2v-1\right| = \ln|x| + c$ ***A1***

$-\frac{1}{2}\ln\left|1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right| - \ln|x| = c$ **OR** $-\frac{1}{2}\ln\left|\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right| - \ln|x| = c$

EITHER

attempt to substitute $x = 2$ and either $y = 0$ or $v = 0$ to find a constant c ***(M1)***

$c = -\ln 2$

$-\frac{1}{2}\ln\left|1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right| - \ln|x| = -\ln 2$ **OR** $-\frac{1}{2}\ln\left|\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right| - \ln|x| = -\ln 2$ ***A1***

OR

$x^2\left|1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right| = A$ **OR** $x^2\left|\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right| = A$ ***A1***

attempt to substitute $x = 2$ either $y = 0$ or $v = 0$ to find a constant A ***(M1)***

THEN

$x^2\left|1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right| = 4$ **OR** $x^2\left(1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right) = 4$ ***A1***

checking boundary values confirms $x^2 - 2xy - y^2 = 4$ ***AG***

Note: Condone absence of absolute value signs even if removed incorrectly until the final ***A1*** mark where they must be seen or have been removed to form a correct equation.

[8 marks]

10. [Maximum mark: 18] 25M.1.AHL.TZ1.10

The function f is defined by $f(x) = 5(x+1)(x+3)$, where $x \in \mathbb{R}$.

(a) Write $f(x)$ in the form $a\left(x-h^2\right)+k$, where $a, h, k \in \mathbb{Z}$. [4]

Markscheme

METHOD 1

$a=5$ ***(A1)***

attempt to use roots and symmetry to find h ***(M1)***

$h=\frac{(-1)+(-3)}{2}$ **OR** half the distance between the roots $\frac{(-1)-(-3)}{2}=1$ (may be seen on a diagram)

$h=-2$ (accept $x=-2$) ***(A1)***

$f(x)=5(x-(-2))^2-5\left(=5(x+2)^2-5\right)$ ***A1***

$(a=5,\ h=-2,\ k=-5)$

METHOD 2

$a=5$ ***(A1)***

attempt to expand

$(x+1)(x+3)=x^2+4x+3$ **OR** $5(x+1)(x+3)=5x^2+20x+15$

EITHER

uses their expansion to attempt to complete the square to the form ***(M1)***

$p(x+q)^2+r$, where q is half the coefficient of their x term

$=(x+2)^2-2^2+3\left(=(x+2)^2-1\right)$ **OR** $5\left[(x+2)^2-2^2+3\right]\left(=5(x+2)^2-5\right)$ ***(A1)***

OR

uses their expansion to attempt to differentiate and sets equal to zero ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=2x+4=0$ **OR** $\frac{\mathrm{d}y}{\mathrm{d}x}=10x+20=0$

$h=-2$ (accept $x=-2$) ***(A1)***

OR

uses their expansion to attempt to find axis of symmetry using $h=\frac{-b}{2a}$ ***(M1)***

$h=\frac{-4}{2}$ **OR** $h=\frac{-20}{10}$

$h=-2$ (accept $x=-2$) ***(A1)***

THEN

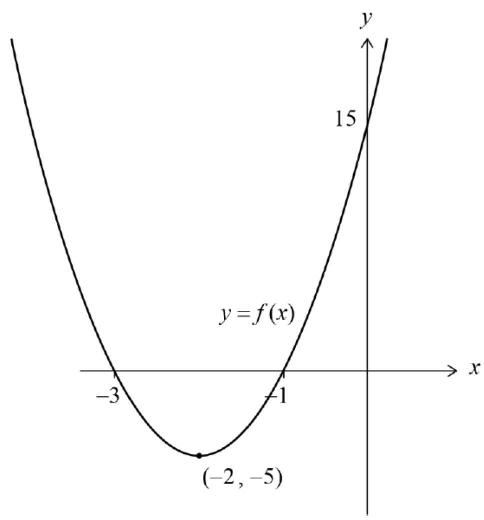
$f(x) = 5(x - (-2))^2 - 5\left(= 5(x+2)^2 - 5\right)$ ***A1***

$(a = 5,\ h = -2,\ k = -5)$

[4 marks]

(b) Sketch the graph of $y = f(x)$, showing the values of any intercepts with the axes and the coordinates of the vertex. [4]

Markscheme



M1A1A1A1

award ***M1*** for a roughly symmetric curve which is concave up

award ***A1*** for x intercepts at -3 and -1

award ***A1*** for y intercept at 15

award ***A1*** for vertex at $(-2, -5)$

[4 marks]

(c) Solve the inequality $f(x) \le 40$. [4]

Markscheme

$5(x+2)^2 - 5 \le 40$ **OR** $5(x+1)(x+3) \le 40$ **OR** $(x+1)(x+3) \le 8$ leading to $(x+2)^2 \le 9$ **OR** $5x^2 + 20x - 25 \le 0$ **OR** $x^2 + 4x - 5 \le 0$ ***(A1)***

valid attempt to find the critical values for their quadratic inequality ***(M1)***

$x + 2 = \pm 3$ **OR** $(x+5)(x-1) = 0$

$x = -5, x = 1$ ***(A1)***

$-5 \le x \le 1$ ***A1***

Note: Accept $(x \in)[-5, 1]$ or equivalent.

[4 marks]

The function g is defined by $g(x) = \ln x$, where $x \in \mathbb{R}, x > 0$.

(d.i) Write down an expression for $(f \circ g)(x)$. [1]

Markscheme

$(f \circ g)(x) = 5(\ln x + 1)(\ln x + 3)$ **OR** $5(\ln x + 2)^2 - 5$ **OR** $5(\ln x)^2 + 20 \ln x + 15$ ***A1***

[1 mark]

(d.ii) Solve the inequality $(f \circ g)(x) \le 40$. [2]

Markscheme

attempt to replace x with $\ln x$ using their solution to part (c) ***(M1)***

$-5 \le \ln x \le 1$

$\mathrm{e}^{-5} \le x \le \mathrm{e}$ ***A1***

Note: Accept $(x \in)\left[\mathrm{e}^{-5}, \mathrm{e}\right]$ or equivalent.

[2 marks]

(e) Find the domain of $g \circ f$. [3]

Markscheme

$(g \circ f)(x) = \ln (f(x))$

recognition that the domain requires $f(x) > 0$ ***(M1)***

$(x + 1)(x + 3) > 0$

$x < -3,\ x > -1$ ***A1A1***

Note: award ***A1*** for critical values and ***A1*** for correct inequalities.
accept $(x \in)(-\infty, -3) \cup (-1, \infty)$ or equivalent.

[3 marks]

11. [Maximum mark: 17] 25M.1.AHL.TZ1.11

The plane Π_1 has equation $x + 2y + z = 0$ and the plane Π_2 has equation $x - y - 2z = 0$.

The acute angle between the planes Π_1 and Π_2 is θ.

(a) Show that $\theta = 60^\circ$. [6]

Markscheme

normals of Π_1 and Π_2 are $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ ***(A1)***

attempt to use the scalar product for the angle between two vectors ***M1***

$$\begin{pmatrix}1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix} = \left|\begin{pmatrix}1\\2\\1\end{pmatrix}\right|\left|\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right|\cos\,\theta$$

$-3 = \sqrt{6}\sqrt{6}\cos\,\theta$ (so $-3 = 6\,\cos\,\theta$) ***A1A1***

Note: Award ***A1*** for correct scalar product and ***A1*** for correct magnitudes of normals and $\cos\,\theta$.

$\cos\,\theta = -\frac{1}{2}$ ***A1***

$\theta = 120^\circ$ **OR** $\cos\,(180^\circ - \theta) = \frac{1}{2}$ ***A1***

acute angle is 60° ***AG***

[6 marks]

A third plane Π_3 is perpendicular to both Π_1 and Π_2.

The unique point of intersection of all three planes is the point $\mathrm{R}(5,\ -5,\ 5)$.

(b) Find the Cartesian equation of Π_3. [4]

Markscheme

attempt to find vector product of their normal vectors ***M1***

$$\begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\-1\\-2\end{pmatrix} = \begin{pmatrix}-4+1\\1+2\\-1-2\end{pmatrix}$$

$$= \begin{pmatrix}-3\\3\\-3\end{pmatrix}\left(= 3\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right)$$ ***(A1)***

equation of Π_3 is $-3x + 3y - 3z = d$ **OR** $-x + y - z = d$ or equivalent

attempt to substitute $x = 5,\ y = -5,\ z = 5$ into their equation ***(M1)***

equation of Π_3 is $-3x + 3y - 3z = -45$ (so $-x + y - z = -15$) ***A1***

[4 marks]

Each of the planes Π_1 and Π_2 contains a mirror.

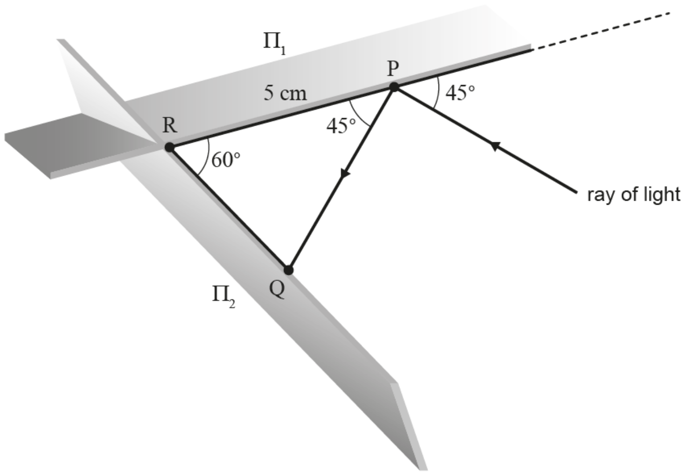
A ray of light is directed towards the mirror in Π_1. The ray of light forms an angle of 45° with Π_1 and meets it at the point P.

The ray of light is then reflected towards the mirror in Π_2, and meets Π_2 at the point Q.

The points P and Q are contained in Π_3.

It is given that $\mathrm{PR} = 5\ \mathrm{cm}$.

This information is shown on the following diagram.



(c.i) Using an appropriate compound angle identity, show that $\sin\ 75^\circ = \frac{\sqrt{2}+\sqrt{6}}{4}$. [3]

Markscheme

attempt to use the identity for $\sin\ (30^\circ + 45^\circ)$ ***(M1)***

$\sin\ (30^\circ + 45^\circ) = \sin\ (30^\circ) \cos\ (45^\circ) + \cos\ (30^\circ) \sin\ (45^\circ)$

$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$ ***A1A1***

$= \frac{\sqrt{2}+\sqrt{6}}{4}$ ***AG***

Note: ***A1*** for each term. Award ***A1 A0*** for correct answers where the denominator has not been rationalized such as $= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$.

[3 marks]

(c.ii) Find QR, giving your answer in the form $p\left(\sqrt{q} - 1\right)\ \mathrm{cm}$ where $p, q \in \mathbb{Z}$. [4]

Markscheme

let $x = \mathrm{QR}$

attempt to use the sine rule with the angles $75^\circ, 45^\circ$ ***(M1)***

$\frac{5}{\sin 75^\circ} = \frac{x}{\sin 45^\circ}$

$x = \frac{20}{\sqrt{2}\left(\sqrt{2}+\sqrt{6}\right)} \left(= \frac{20}{2\left(1+\sqrt{3}\right)}\right)$ or equivalent ***A1***

attempt to rationalise a denominator of the form $\left(\sqrt{a} + \sqrt{b}\right)$ multiplying numerator and denominator by $\pm\left(\sqrt{a} - \sqrt{b}\right)$ or equivalent ***(M1)***

$x = \frac{20\left(\sqrt{2}-\sqrt{6}\right)}{\sqrt{2}\left(\sqrt{2}+\sqrt{6}\right)\left(\sqrt{2}-\sqrt{6}\right)} \left(= \frac{20\left(1-\sqrt{3}\right)}{2\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)}\right)$

$x = 5\left(\sqrt{3} - 1\right)$ (cm) $(p = 5,\ q = 3)$ ***A1***

[4 marks]

12. [Maximum mark: 19] 25M.1.AHL.TZ1.12

Consider the family of functions $f_n(x) = \cos^n x$, where $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

(a) By writing $\cos^n x$ as $\cos^{n-1} x \cos x$, show that $\int \cos^n x \text{ d } x = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \text{ d } x - (n-1) \int \cos^n x \text{ d } x$ for $n > 1$. [4]

Markscheme

attempt to use integration by parts on $f_n(x) = \cos^{n-1} x \cos x$ ***M1***

$u = \cos^{n-1} x$, $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = -(n-1)\cos^{n-2} x \sin x, v = \sin x$

$\int f_n(x) \text{ d } x = \cos^{n-1} x \sin x + \int (n-1)\cos^{n-2} x \sin^2 x \text{ d } x$ ***A1A1***

Note: ***A1*** for each term with correct signs

$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \left(1 - \cos^2 x\right) \text{ d } x$ ***A1***

$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \text{ d } x - (n-1) \int \cos^n x \text{ d } x$ ***AG***

[4 marks]

(b) Hence, show that $\int f_n(x) \text{ d } x = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} \int f_{n-2}(x) \text{ d } x$ for $n > 1$. [2]

Markscheme

attempt to rearrange the equation given in part (a) to collect terms in $\int f_n(x) \text{ d } x$ or $\int \cos^n x \text{ d } x$ ***M1***

$\int f_n(x)\ \mathrm{d}\ x + (n-1)\int f_n(x)\ \mathrm{d}\ x = \cos^{n-1}\ x\ \sin\ x + (n-1)\int f_{n-2}(x)\ \mathrm{d}\ x$ **OR**
$\int \cos^n\ x\ \mathrm{d}\ x + (n-1)\int \cos^n\ x\ \mathrm{d}\ x = \cos^{n-1}\ x\ \sin\ x + (n-1)\int \cos^{n-2}\ x\ \mathrm{d}\ x$

$n\int f_n(x)\ \mathrm{d}\ x = \cos^{n-1}\ x\ \sin\ x + (n-1)\int f_{n-2}(x)\ \mathrm{d}\ x$ **OR**
$n\int \cos^n\ x\ \mathrm{d}\ x = \cos^{n-1}\ x\ \sin\ x + (n-1)\int \cos^{n-2}\ x\ \mathrm{d}\ x$ ***A1***

$\int f_n(x)\ \mathrm{d}\ x = \frac{1}{n}\cos^{n-1}\ x\ \sin\ x + \frac{n-1}{n}\int f_{n-2}(x)\ \mathrm{d}\ x$ ***AG***

[2 marks]

(c) Hence, find an expression for $\int \cos^4\ x\ \mathrm{d}\ x$, giving your answer in the form $p\ \cos^3\ x\ \sin\ x + q\ \cos\ x\ \sin\ x + rx + c$ where $p,\ q,\ r\ \in\ \mathbb{Q}^+$. [4]

Markscheme

attempt to use equation from part (a) to reduce the power of $\cos\ x$ ***(M1)***

$\int \cos^4\ x\ \mathrm{d}\ x = \frac{1}{4}\cos^3\ x\ \sin\ x + \frac{3}{4}\int \cos^2\ x\ \mathrm{d}\ x$

EITHER

attempt to use equation from part (a) again to reduce the power of $\cos\ x$ ***(M1)***

$\int \cos^4\ x\ \mathrm{d}\ x = \frac{1}{4}\cos^3\ x\ \sin\ x + \frac{3}{4}\left(\frac{1}{2}\cos\ x\ \sin\ x + \frac{1}{2}\int 1\ \mathrm{d}\ x\right)$ ***(A1)***

OR

attempt to use double angle formula to rewrite $\cos^2\ x$ in terms of $\cos\ 2x$ ***(M1)***

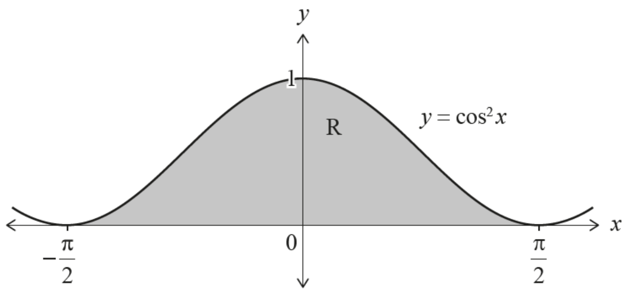
$\int \cos^4\ x\ \mathrm{d}\ x = \frac{1}{4}\cos^3\ x\ \sin\ x + \frac{3}{4}\int\left(\frac{\cos\ 2x+1}{2}\right)\ \mathrm{d}\ x$ ***(A1)***

THEN

$\frac{1}{4}\cos^3\ x\ \sin\ x + \frac{3}{8}\cos\ x\ \sin\ x + \frac{3}{8}x + c$ ***A1***

[4 marks]

The region R is enclosed by the graph of $y = \cos^2\ x$ and the x-axis where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, as shown in the following diagram.



The region R is rotated by 2π radians around the x-axis to form a solid of revolution.

(d) Find the volume of the solid. [4]

Markscheme

attempt to use the formula for volume of revolution using π and $\left(\cos^2 x\right)^2$ ***(M1)***

volume $= \int \pi\left(\cos^2 \mathrm{x}\right)^2 \mathrm{d}\ \mathrm{x}$

volume $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi\left(\cos^2 x\right)^2 \mathrm{d}\ x$ **OR** $2\int_{0}^{\frac{\pi}{2}} \pi\left(\cos^2 x\right)^2 \mathrm{d}\ x$ ***(A1)***

Note: Condone omission $\mathrm{d}\ x$ for the ***A1***.

attempt to substitute correct limits into their (c) and subtract ***(M1)***

$\pi\left(\frac{1}{4}\cos^3 \frac{\pi}{2}\sin \frac{\pi}{2} + \frac{3}{8}\cos \frac{\pi}{2}\sin \frac{\pi}{2} + \frac{3}{8}\left(\frac{\pi}{2}\right)\right) - \pi\left(\frac{1}{4}\cos^3 \left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + \frac{3}{8}\cos\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + \frac{3}{8}\left(-\frac{\pi}{2}\right)\right)$

$= \frac{3\pi^2}{8}$ ***A1***

Note: ***FT*** marks may be awarded for a final answer of $r\pi^2$ based on non-zero values of p, q and r.

[4 marks]

(e.i) Find the Maclaurin series of $f_n(x)$ up to the term in x^2. [3]

Markscheme

METHOD 1

attempt to raise Maclaurin expansion for $\cos\ x$ to the power of n ***(M1)***

$f_n(x) = (\cos\ x)^n = \left(1 - \frac{x^2}{2} + \ldots\right)^n$

$= 1 - \frac{nx^2}{2} + \ldots$ ***A2***

METHOD 2

attempt to differentiate $f_n(x)$ twice ***(M1)***

$f_n{\prime}(x) = -n\cos^{n-1}x\sin x,\ f_n{\prime\prime}(x) = -n\cos^n x + n(n-1)\cos^{n-2}x\sin^2 x$ ***A1***

$f_n(0) = 1,\ f_n{\prime}(0) = 0,\ f_n{\prime\prime}(0) = -n$

$f_n(x) = 1 - \frac{nx^2}{2} + \ldots$ ***A1***

[3 marks]

(e.ii) Hence or otherwise, find $\lim\limits_{x\to 0}\frac{f_n(x)-1}{x^2}$ in terms of n. [2]

Markscheme

METHOD 1

attempt to use Maclaurin expansion for $f_n(x)$ ***(M1)***

$\lim\limits_{x\to 0}\frac{f_n(x)-1}{x^2} = \lim\limits_{x\to 0}\frac{-\frac{nx^2}{2}+\ldots}{x^2}\left(= \lim\limits_{x\to 0}\left(-\frac{n}{2} + \text{ powers of } x\right)\right)$

$= -\frac{n}{2}$ ***A1***

METHOD 2

attempt to use l'Hôpital's rule twice on $\frac{\cos^n(x)-1}{x^2}$ ***(M1)***

$\lim\limits_{x\to 0}\frac{\cos^n(x)-1}{x^2} = \lim\limits_{x\to 0}\frac{-n\cos^{n-1}x\sin x}{2x} = \lim\limits_{x\to 0}\frac{n(n-1)\cos^{n-2}x\sin^2 x - n\cos^n x}{2}$

$= -\frac{n}{2}$ ***A1***

Note: Do not award ***FT*** marks for an expression that does not involve n.

[2 marks]