INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

Math AA

EXERCISES [Math-AA 3.9]

FURTHER TRIGONOMETRIC FUNCTIONS

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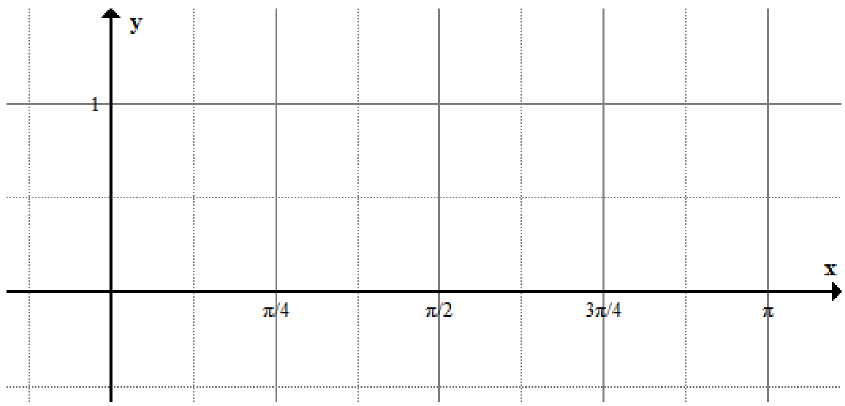
O. Practice questions

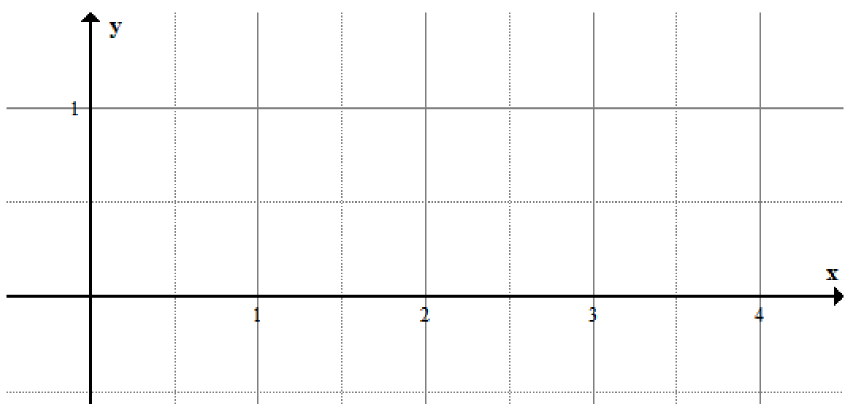
1. [Maximum mark: 12] ***[without GDC]***

(a) Sketch the graph of the function $f(x) = |\sin 4x|$, $0 \le x \le \pi$



[4]

(b) Sketch the graph of the function $g(x) = |\sin(\pi x)|$, $0 \le x \le 4$



[4]

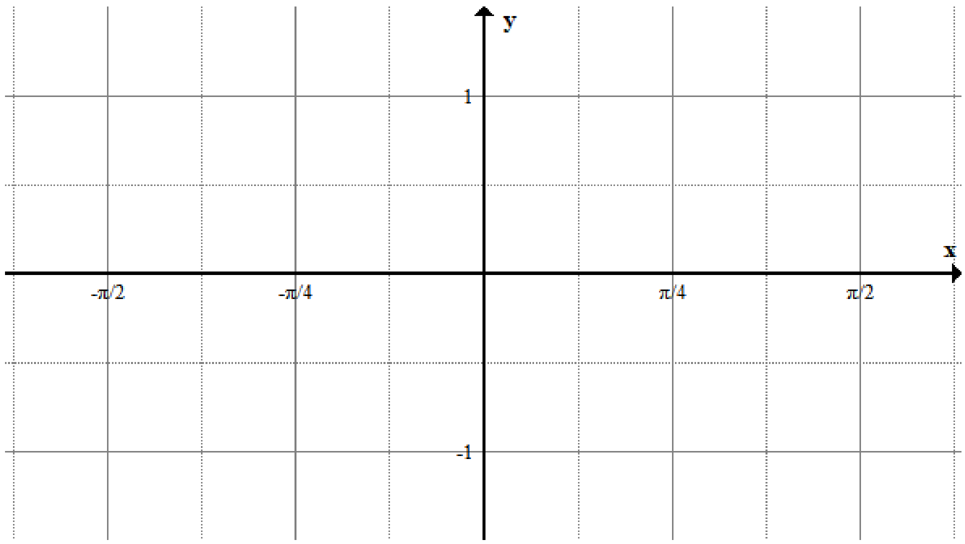
(c) Based on the graphs, solve the following equations

$\lvert\sin 4x\rvert = 1$, $0 \le x \le \pi$	
$\lvert\sin(\pi x)\rvert = 1$, $0 \le x \le 4$	

[4]

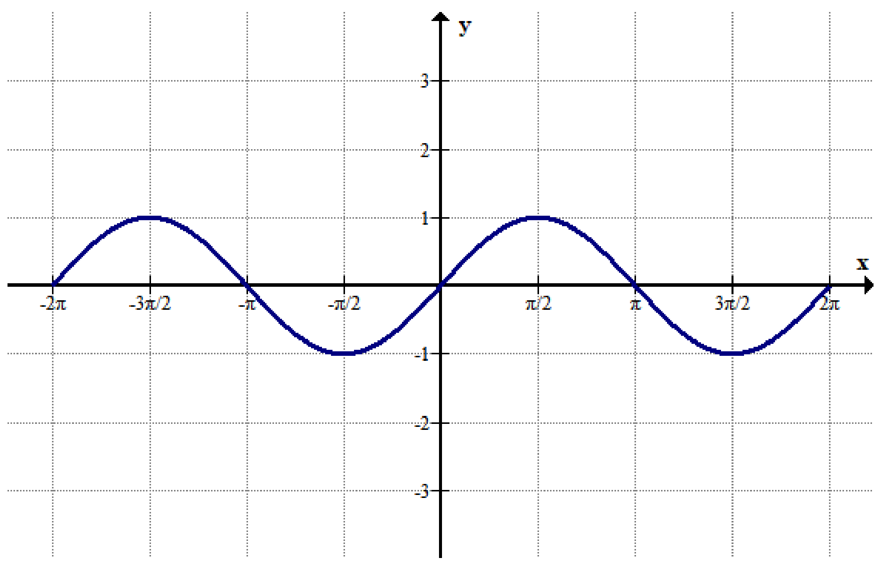
2. [Maximum mark: 4] ***[without GDC]***

Sketch the graphs of the function $f(x)=\sin|4x|$, $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$



3. [Maximum mark: 6] ***[without GDC]***

The diagram shows part of the graph of $y=\sin x$.



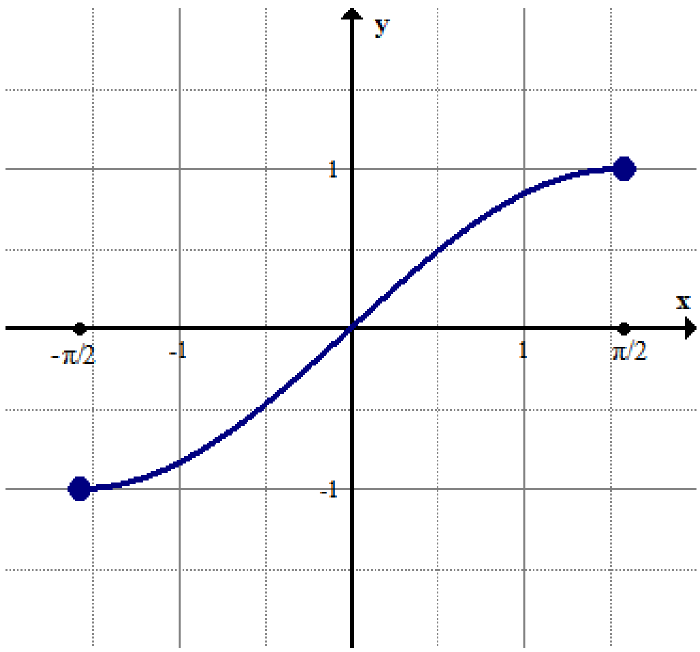
(a) On the same diagram, sketch the graph of $y=\operatorname{cosec} x=\frac{1}{\sin x}$. [4]

(b) Write down the domain and the range of $y=\operatorname{cosec} x$. [2]

4. [Maximum mark: 5] ***[without GDC]***

The graph of $f(x)=\sin x$, $-\frac{\pi}{2}\leq x\leq\frac{\pi}{2}$ is shown below.

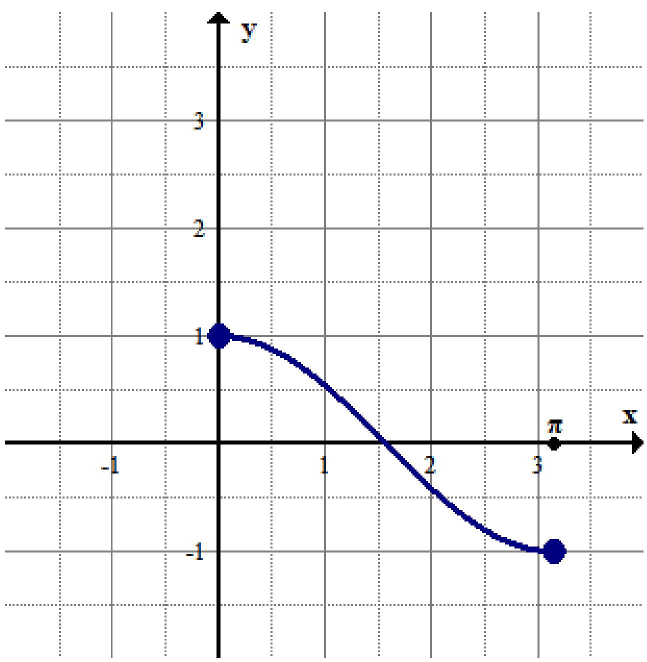
On the same diagram sketch the graph of $f^{-1}(x)=\arcsin x$



5. [Maximum mark: 5] ***[without GDC]***

The graph of $f(x)=\cos x$, $0\leq x\leq\pi$ is shown below.

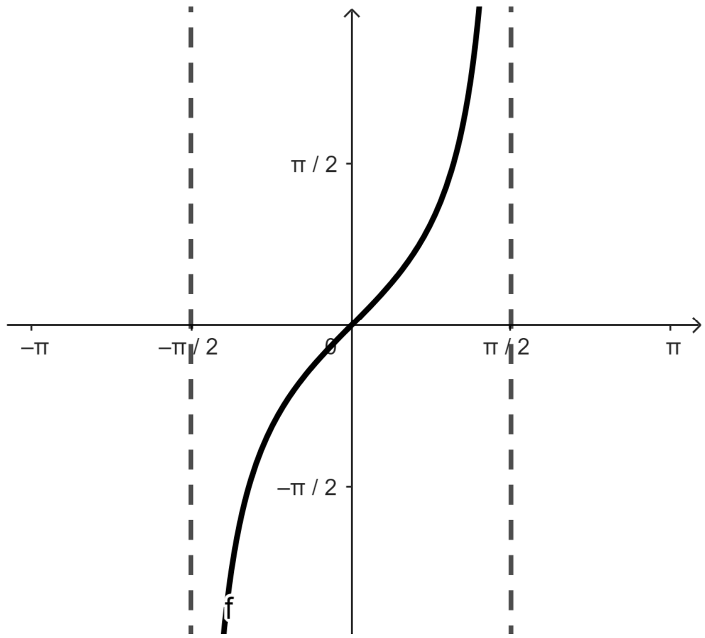
On the same diagram sketch the graph of $f^{-1}(x)=\arccos x$



6. [Maximum mark: 5] ***[without GDC]***

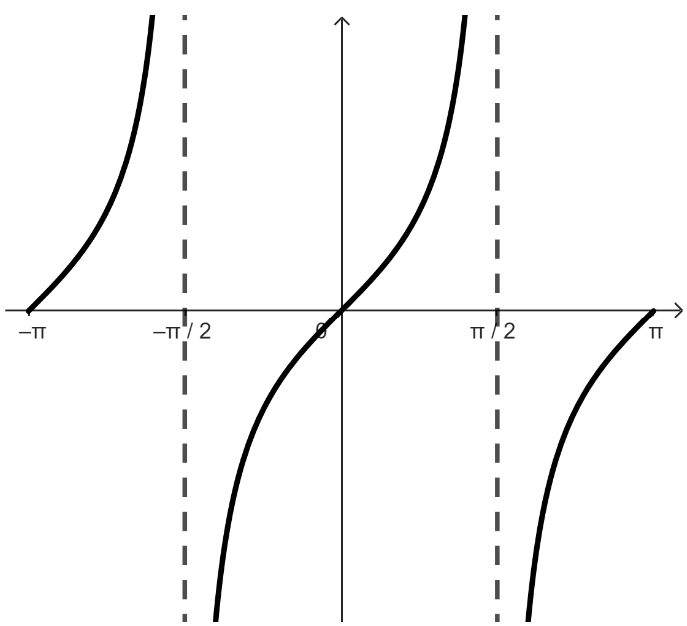
(a) The graph of $f(x)=\tan x$, $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ is shown below.

On the same diagram sketch the graph of $f^{-1}(x)=\arctan x$



(b) The graph of $f(x)=\tan x$, $-\pi\le x\le\pi$ is shown below.

On the same diagram sketch the graph of $\frac{1}{f(x)}=\cot x$, $-\pi\le x\le\pi$



7. [Maximum mark: 9] ***[without GDC]***

By sketching an appropriate right-angle triangle, or otherwise, find the values of

(i) $\sin\left(\arcsin\dfrac{2}{3}\right)$, (ii) $\tan\left(\arcsin\dfrac{2}{3}\right)$, (ii) $\cos\left(2\arcsin\dfrac{2}{3}\right)$.

8. [Maximum mark: 6] ***[without GDC]***

Find the value of $\sin\left(\arccos\dfrac{2}{3}+\arctan\dfrac{4}{3}\right)$,

9. [Maximum mark: 14] ***[without GDC]***

(a) Write down the domain and the range of the function $f(x)=\arctan x$. [2]

(b) Find the values of

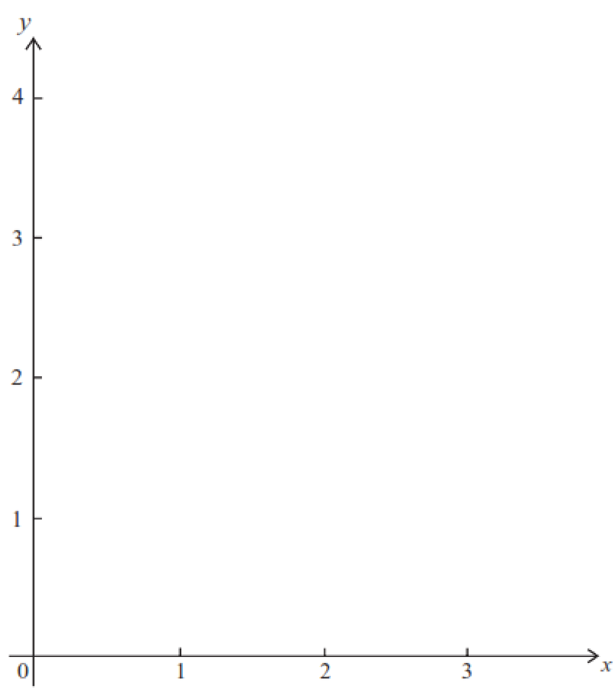
$A=\arctan\dfrac{1}{3}+\arctan\dfrac{1}{2}$, $B=\arctan 2+\arctan 3$

$C=\arctan\dfrac{1}{3}-\arctan 2$, $D=\arctan\dfrac{2}{3}+\arctan\dfrac{3}{2}$ [12]

A. Exam style questions (SHORT)

10. [Maximum mark: 7] ***[without GDC]***

(a) Sketch the curve $f(x)=\left|1+3\sin(2x)\right|$, for $0\le x\le\pi$. Write down on the graph the values of the x and y intercepts. [4]



(b) By adding **one** suitable line to your sketch, find the number of solutions to the equation $\pi f(x)=4(\pi-x)$. [3]

11. [Maximum mark: 10] ***[without GDC]***

(a) Sketch the graph of $y = \cos(4x)$, in the interval $0 \le x \le \pi$. [3]

(b) On the same diagram sketch the graph of $y = \sec(4x)$, for $0 \le x \le \pi$, indicating clearly the equations of any asymptotes. [3]

(c) Use your sketch to solve

(i) the equation $\sec(4x) = -1$, for $0 \le x \le \pi$.

(ii) the inequality $\sec(4x) < 0$, for $0 \le x \le \pi$. [4]

12. [Maximum mark: 10] ***[without GDC]***

Consider the function $f(x) = \left|\csc(\pi x)\right|$.

(a) Sketch the graph of $y = \sin(\pi x)$, in the interval $0 \le x \le 4$. [3]

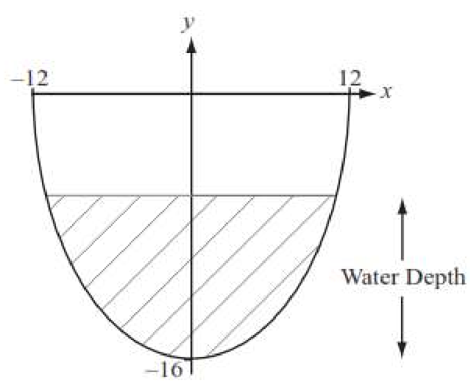
(b) On the same diagram sketch the graph of $y = f(x)$, for $0 \le x \le 4$, by indicating clearly the equations of any asymptotes. [4]

(c) Solve the equation $f(x) = 1$, for $0 \le x \le 4$. [2]

(d) Write down the number of solutions of the equation $f(x) = 2$, for $0 \le x \le 4$. [1]

13. [Maximum mark: 6] ***[without GDC]***

The diagram below shows the boundary of the cross-section of a water channel.



The equation that represents this boundary is $y = 16\sec\left(\frac{\pi x}{36}\right) - 32$, where x and y are both measured in cm. The top of the channel is level with the ground and has a width of $24\ \text{cm}$. The maximum depth of the channel is $16\ \text{cm}$.

Find the width of the water surface in the channel when the water depth is $10\ \text{cm}$. Give your answer in the form $a \arccos b$ where $a, b \in \mathbb{R}$

14. [Maximum mark: 5] ***[without GDC]***

Solve the equation $\cos(2\arcsin x) = 0.68$

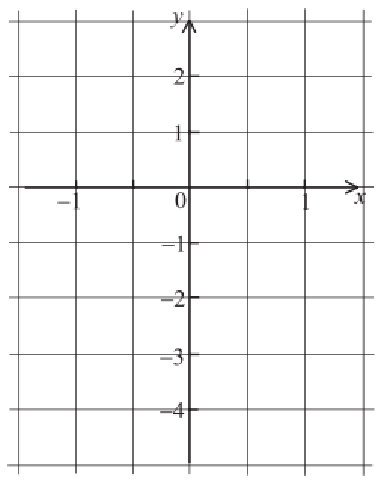
15. [Maximum mark: 7] ***[without GDC]***

Solve the equation $\arctan x + \arctan 2x = \arctan\sqrt{2}$

16. [Maximum mark: 6] ***[with GDC]***

Let $f(x) = x\arccos x + \frac{1}{2}x$ for $-1 \le x \le 1$ and $g(x) = \cos 2x$ for $-1 \le x \le 1$.

(a) On the grid below, sketch the graph of f and of g. [3]



(b) Write down the solution of the equation $f(x) = g(x)$. [1]

(c) Write down the range of g. [2]

17. [Maximum mark: 6] ***[with GDC]***

A system of equations is given by

$$\cos x + \cos y = 1.2 \qquad \sin x + \sin y = 1.4$$

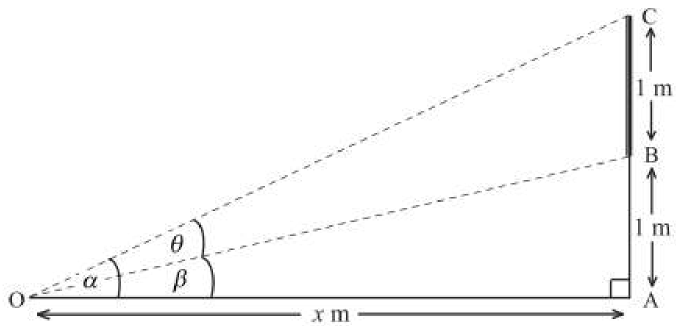
(a) For each equations express y in terms of x. [3]

(b) **Hence** solve the system for $0 < x < \pi$, $0 < y < \pi$. [3]

B. Exam style questions (LONG)

18. [Maximum mark: 12] ***[with GDC]***

A television screen, BC, of height one metre, is built into the wall. The bottom of the television screen at B is one metre above an observer's eye level. The angles of elevation ($\mathrm{A\hat{O}C}$, $\mathrm{A\hat{O}B}$) from the observer's eye at O to the top and bottom of the television screen are α and β radians respectively. The horizontal distance from the observer's eye to the wall containing the television screen is x metres. The observer's angle of vision ($\mathrm{B\hat{O}C}$) is θ radians, as shown below.



(a) Show that $\theta = \arctan\frac{2}{x} - \arctan\frac{1}{x}$. [4]

(b) Express θ in the form $\theta = \arctan\frac{ax}{x^2+b}$, where $a, b \in R$. [3]

(c) Find the maximum value of angle θ, and the value of x where this max occurs. [2]

(d) Find where the observer should stand so that the angle of vision is 15°. [3]

19. [Maximum mark: 15] ***[with GDC]***

The function f is defined by $f(x) = \operatorname{cosec} x + \tan 2x$.

(a) Sketch the graph of f for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, and state

(i) the x-intercepts;

(ii) the equations of the asymptotes;

(iii) the coordinates of the maximum and minimum points. [6]

(b) Show that the roots of $f(x) = 0$ satisfy the equation

$$2\cos^3 x - 2\cos^2 x - 2\cos x + 1 = 0$$ [5]

(c) Show that $f(\pi - x) + f(\pi + x) = 0$ [4]

20. [Maximum mark: 21] ***[without GDC]***

(a) Sketch the curve $f(x)=\sin 2x$, $0 \le x \le \pi$. [2]

(b) On a separate diagram, sketch the graph of $g(x)=\operatorname{cosec} 2x$, $0 \le x \le \pi$. State the coordinates of any local max or min points and the equations of any asymptotes. [5]

(c) Show that $\tan x+\cot x=2\operatorname{cosec} 2x$. [3]

(d) Hence or otherwise, find the coordinates of the local maximum and local minimum points on the graph of $y=\tan 2x+\cot 2x$, $0 \le x \le \frac{\pi}{2}$. [5]

(e) Find the solution of the equation $\operatorname{cosec} 2x=1.5\tan x-0.5$, $0 \le x \le \frac{\pi}{2}$. [6]

21*. [Maximum mark: 11] ***[with GDC]***

Consider the function $C(x)=\cos x+\frac{1}{2}\cos 2x$ for $-2\pi \le x \le 2\pi$. By observing the graph of the function in the given domain

(a) State its period T and prove that $C(x)$ is periodic , that is $C(x+T)=C(x)$. [3]

(b) For what values of x, $-2\pi \le x \le 2\pi$, is $C(x)$ a maximum? [2]

(c) Let $x=x_0$ be the smallest positive value of x for which $C(x)=0$. Find an approximate value of x_0 which is correct to two significant figures. [2]

(d) (i) Prove that $C(x)=C(-x)$ for all x.

(ii) Let $x=x_1$ be that value of x, $\pi < x < 2\pi$, for which $C(x)=0$. Find the value of x_1 in terms of x_0. [4]

22*. [Maximum mark: 12] ***[without GDC]***

The function f is defined by $f(x)=\cos x+\sqrt{3}\sin x$. This function may also be expressed in the form $R\cos(x-\alpha)$ where $R>0$ and $0<a<\frac{\pi}{2}$.

(a) By expanding $R\cos(x-\alpha)$ and comparing with the original form of f, find the **exact** value of R and of α. [5]

(b) State (i) the period of f. (ii) the range of f. [3]

(c) Find the **exact** value of x in $\left[0, \frac{\pi}{2}\right]$ satisfying the equation $f(x)=\sqrt{2}$. [4]

23.** [Maximum mark: 19] ***[without GDC]***

(a) Show that $\cos(A+B)+\cos(A-B)=2\cos A\cos B$. [2]

(b) Let $T_n(x)=\cos(n\arccos x)$ where x is a real number, $x\in[-1,1]$ and $n\in\mathbb{Z}^+$.

(i) Find $T_1(x)$.

(ii) Show that $T_2(x)=2x^2-1$. [5]

(c) (i) Use the result in part (a) to show that $T_{n+1}(x)+T_{n-1}(x)=2xT_n(x)$

(ii) Prove by mathematical induction that $T_n(x)$ is a polynomial of degree n. [12]

24. [Maximum mark: 16] ***[without GDC]***

(a) Find the value of $\tan(\arccos(0.1))$ [4]

(b) Find the value of $\sin(2\arctan 5)$ [5]

(c) (i) Find the value of $\sin\left(\arctan\frac{1}{2}+\arctan\frac{1}{3}\right)$ by using the identity for $\sin(A+B)$

(ii) Deduce the value of $\arctan\frac{1}{2}+\arctan\frac{1}{3}$. [7]