INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

Math AA

EXERCISES [Math-AA 3.8]

FURTHER TRIGONOMETRIC IDENTITES AND EQUATIONS

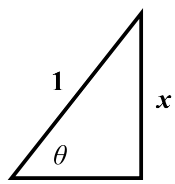
Compiled by Christos Nikolaidis

O. Practice questions

TRIGONOMETRIC IDENTITIES

1. [Maximum mark: 6] ***[without GDC]***

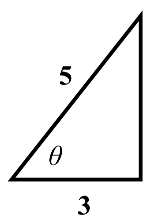
It is given that $\sin\theta = x$ where $0° < \theta < 90°$. Use the following triangle,



to find $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$ in terms of x.

2. [Maximum mark: 6] ***[without GDC]***

It is given that $\cos\theta = -\frac{3}{5}$ where $90° < \theta < 180°$. Use the following triangle,



and the fact that θ is in the 2nd quadrant to find $\sin\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$.

[**Notice**: Don't bother for the sign (–) on the diagram. Find the absolute values first, using the triangle, and then decide if the results are + or – according to the quadrant]

3. [Maximum mark: 12] ***[without GDC]***

It is given that $\tan\theta = 3$ where θ is an angle in the first quadrant.

(a) Use an appropriate Pythagorean trigonometric identity to find $\sec\theta$. [3]

(b) **Hence** find $\cos\theta$, $\sin\theta$. [3]

(c) Sketch a right-angled triangle to represent $\tan\theta = 3$ and hence find $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$. [6]

4. [Maximum mark: 15] ***[without GDC]***

(a) Show that $\sin 15^o = \frac{\sqrt{6}-\sqrt{2}}{4}$ by using the trigonometric identity for $\sin(A-B)$. [3]

(b) Find a similar expression for $\cos 15^o$. [3]

(c) Show that $\tan 15^o = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

(i) using the results above (ii) using the identity for $\tan(A-B)$ [5]

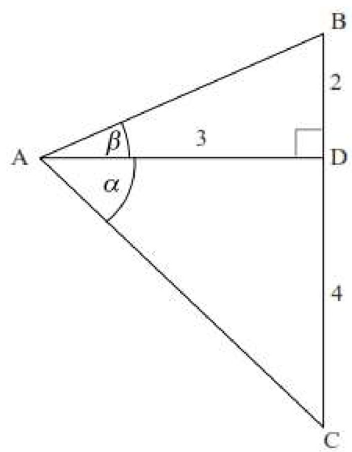
(d) Express $\tan 15^o$ in the form $a-\sqrt{b}$, where a and b are integers. [2]

(e) Express $\cot 15^o$ in the form $c+\sqrt{d}$, where c and d are integers. [2]

5. [Maximum mark: 12] ***[without GDC]***

In the diagram below, AD is perpendicular to BC, CD = 4, BD = 2 and AD = 3

$\mathrm{C\hat{A}D} = \alpha$ and $\mathrm{B\hat{A}D} = \beta$.



(a) Find the exact value of $\cos(\alpha+\beta)$

(i) by using the cosine rule in triangle ABC.

(ii) by using the compound angle identity for $\cos(\alpha+\beta)$ [8]

(b) Find the exact value of $\sin(\alpha+\beta)$. [4]

TRIGONOMETRIC EQUATIONS

6. [Maximum mark: 9] ***[without GDC]***

Find the **general solutions** (in radians) of the equations

$\sec x = 2$	
$\operatorname{cosec} x = 2$	
$\cot x = \sqrt{3}$	

7. [Maximum mark: 6] ***[without GDC]***

Solve the equation $\sec^2 x = 2$, $0 \le x \le 2\pi$

8. [Maximum mark: 12] ***[without GDC]***

It is given that x satisfies the equation $\sin(x+\frac{\pi}{3}) = 2\cos(x-\frac{\pi}{6})$.

(a) Find the values of $\tan x$ and $\cot x$. [5]

(b) Find the possible values of $\cos x$. [3]

(c) Solve the equation $\sin(x+\frac{\pi}{3}) = 2\cos(x-\frac{\pi}{6})$ for $-\pi \le x \le \pi$. [4]

9. [Maximum mark: 12] ***[without GDC]***

(a) Solve the equation $\sin 5x = \sin 3x$, $0 \le x \le \pi$. [6]

(b) Explain why $\sin(-x) = -\sin x$ [1]

(c) **Hence**, solve the equation $\sin 5x + \sin 3x = 0$, $0 \le x \le \pi$. [5]

10. [Maximum mark: 12] ***[without GDC]***

(a) Solve the equation $\cos 5x = \cos 3x$, $0 \le x \le \pi$. [6]

(b) Explain why $\cos(\pi - x) = -\cos x$ [1]

(c) **Hence**, solve the equation $\cos 5x + \cos 3x = 0$, $0 \le x \le \pi$. [5]

11. [Maximum mark: 10] ***[without GDC]***

(a) Solve the equation $\tan 5x = \tan 3x$, $0 \le x \le \pi$. [4]

(b) Explain why $\tan(-x) = -\tan x$ [1]

(c) **Hence**, solve the equation $\tan 5x + \tan 3x = 0$, $0 \le x \le \frac{\pi}{2}$. [4]

12. [Maximum mark: 12] ***[without GDC]***

Solve the equations

(a) $\cos 5x = \sin 3x$, $0 \le x \le \pi$. [6]

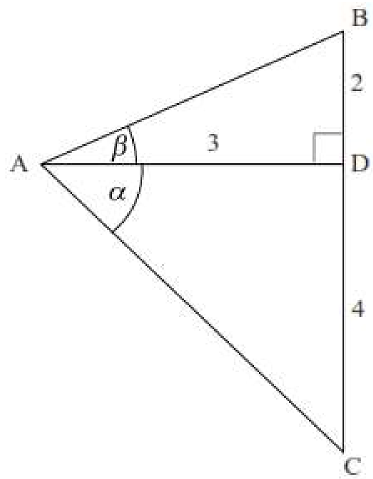
(b) $\cos 5x + \sin 3x = 0$, $0 \le x \le \pi$. [6]

A. Exam style questions (SHORT)

13. [Maximum mark: 9] ***[without GDC]***

In the diagram below, AD is perpendicular to BC, CD = 4, BD = 2 and AD = 3

$C\hat{A}D = \alpha$ and $B\hat{A}D = \beta$.

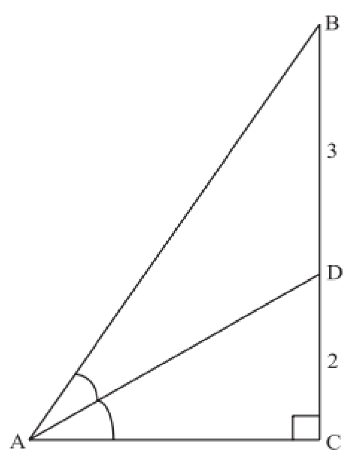


(a) Find the exact value of $\sin(2\beta)$ [4]

(b) Find the exact value of $\cos(\alpha - \beta)$. [5]

14. [Maximum mark: 7] ***[with GDC]***

Let ABC be the right-angled triangle, where $\hat{C}$ = 90. The line (AD) bisects $B\hat{A}C$, BD = 3 and DC = 2, as shown in the diagram. Let $D\hat{A}C$



(a) Find the exact value of $\tan D\hat{A}C$. [6]

(b) Hence, or otherwise, find $D\hat{A}C$. [1]

15. [Maximum mark: 8] ***[without GDC]***

(a) Use the formula for $\cos(A+B)$ to show that

$$4\cos(x+30°)\cos(x+60°) \equiv \sqrt{3} - 2\sin 2x$$ [5]

(b) **Hence** find the exact value of $\cos(75°)\cos(105°)$ [3]

16. [Maximum mark: 10] ***[without GDC]***

(a) Show that

$$\cos\left(x+\frac{\pi}{6}\right)\cos\left(x+\frac{\pi}{3}\right) \equiv \frac{\sqrt{3}-2\sin 2x}{4}$$ [5]

(b) **Hence** solve the equation

$$4\cos\left(x+\frac{\pi}{6}\right)\cos\left(x+\frac{\pi}{3}\right) = \sqrt{3}-1, \quad 0 \le x \le 2\pi$$ [5]

17*. [Maximum mark: 6] ***[without GDC]***

Let $\hat{A}, \hat{B}, \hat{C}$ be the angles of a triangle. Use the fact $\hat{A}+\hat{B}+\hat{C} = 180^o$ to show that

$$\tan\hat{A} + \tan\hat{B} + \tan\hat{C} = \tan\hat{A}\tan\hat{B}\tan\hat{C}.$$

18. [Maximum mark: 6] ***[without GDC]***

Solve the equation $3\operatorname{cosec}^2 x = 4$, $0 \le x \le 2\pi$

19. [Maximum mark: 6] ***[without GDC]***

Solve the equation $\cos 3x = \cos(0.5x)$, for $0 \le x \le \pi$.

20. [Maximum mark: 6] ***[without GDC]***

Solve the equation $\sin x = \sin\frac{x}{3}$, for $0 \le x \le 4\pi$.

21. [Maximum mark: 6] ***[without GDC]***

Solve the equation $\cos(2x-15°) = \cos x$, for $0° \le x \le 360°$.

22. [Maximum mark: 6] ***[without GDC]***

Solve the equation $\tan\left(3\pi x - \frac{\pi}{4}\right) = \tan(\pi x)$, for $0 \le x \le 2$.

23. [Maximum mark: 6] ***[without GDC]***

The angle θ satisfies the equation $2\tan^2\theta - 5\sec\theta - 10 = 0$, where θ is in the second quadrant. Find the **exact** value of $\sec\theta$.

24. [Maximum mark: 6] ***[with GDC]***

The angle θ satisfies the equation $\tan\theta + \cot\theta = 3$.

(a) Find all the possible values of $\tan\theta$

(b) Hence, or otherwise, find all the possible values of θ in $[0°, 90°]$.

25. [Maximum mark: 6] ***[without GDC]***

Solve the equation $\tan x + \cot x = 2$, $0 \le x \le \frac{\pi}{2}$

26*. [Maximum mark: 6] ***[without GDC]***

Solve the equation $\tan x + \cot x = 4$, $0 \le x \le \frac{\pi}{2}$

B. Exam style questions (LONG)

27*. [Maximum mark: 22] ***[without GDC]***

(a) Using the formula for $\cos(A+B)$ and suitable double angle identities, prove the following triple angle identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$ [4]

(b) Find a similar expression for $\sin 3\theta$ in terms of $\sin\theta$ only. [5]

(c) Show that $\tan 3\theta \equiv \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ [5]

(d) Find an expression for $\cos 5\theta$ in terms of $\sin\theta$ and $\cos\theta$. [5]

(e) Hence show that $\dfrac{\cos 5\theta}{\cos\theta} = 16\sin^4\theta - 12\sin^2\theta + 1$ [3]

28. [Maximum mark: 15] ***[without GDC]***

(a) Show that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$ [4]

(b) Solve the equation $4\cos^3\theta - 3\cos\theta + 1 = 0$, for $0 \le \theta \le 2\pi$ [5]

(c) Consider the equation $\cos 3\theta = 11\cos^2\theta$

(i) Find the three values of $\cos\theta$ that satisfy the equation

(ii) Hence, solve the equation for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. [6]

INDUCTION

Please look at also the **TRIGONOMETRY** section from the **EXERCISES**

[MAA 1.9] MATHEMATICAL INDUCTION