INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

Math AA

EXERCISES [Math-AA 3.7]

***TRIGONOMETRIC* FUNCTIONS**

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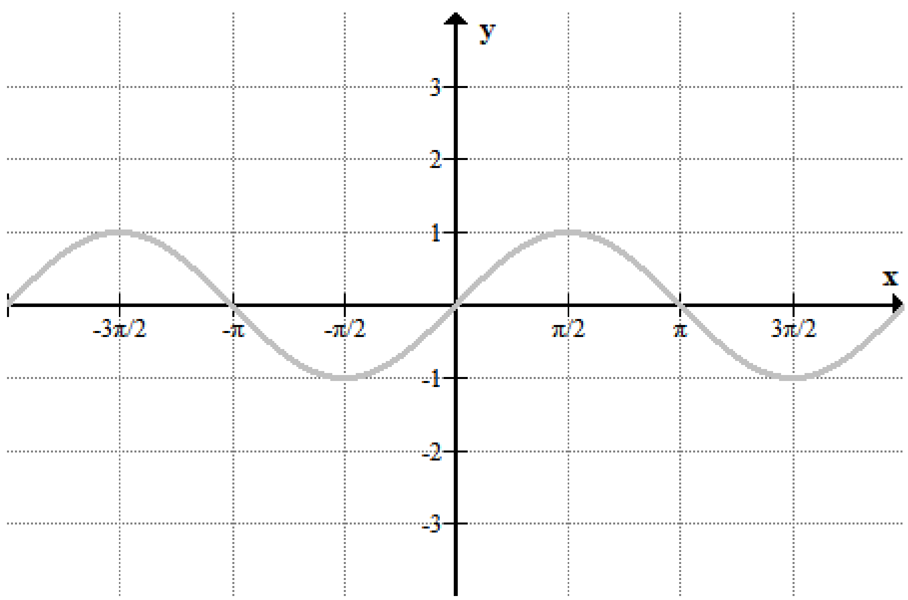
O. Practice questions

1. [Maximum mark: 9] ***[without GDC]***

The following diagram shows part of the graph of $y = \sin x$



(a) On the same diagram sketch the graphs of the functions

(i) $f(x) = \sin x + 2$ (ii) $g(x) = \sin x + 2$ [4]

(b) Complete the following table.

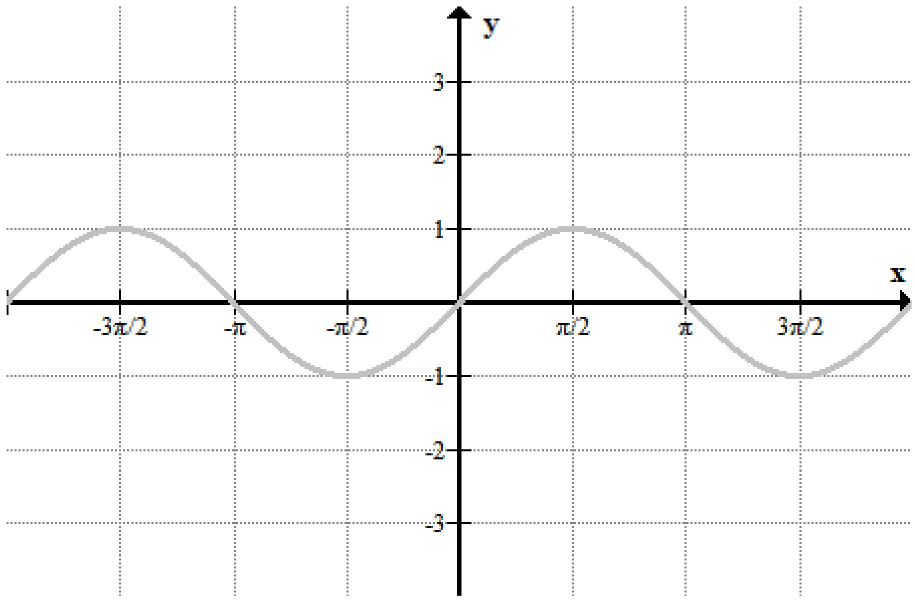
Function	Amplitude	Period	Central axis	Range
$y = \sin x$	1	2π	$y = 0$	$-1 \le y \le 1$
$f(x) = \sin x + 2$				
$g(x) = \sin x - 2$				

[4]

(c) Write down the equation of the **central axis** (midline) for $f(x) = \sin x + c$. [1]

2. [Maximum mark: 9] ***[without GDC]***

The following diagram shows part of the graph of $y = \sin x$



(a) On the same diagram sketch the graph of the function $f(x) = 3\sin x$. [2]

(b) Complete the following table.

Function	Amplitude	Period	Central axis	Range
$y = \sin x$	1	2π	$y = 0$	$-1 \le y \le 1$
$f(x) = 3\sin x$				
$f(x) = -3\sin x$				

[4]

(c) Write down the **amplitude** for the function $f(x) = a\sin x$. [1]

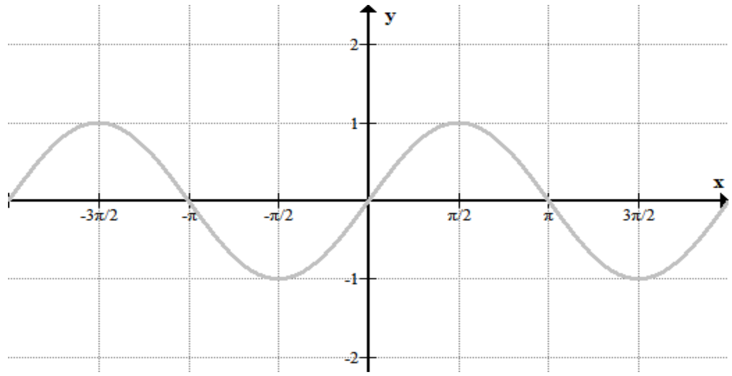
(d) For the function $f(x) = a\sin x + c$, write down

(i) the amplitude (ii) the equation of the central axis. [2]

3. [Maximum mark: 12] ***[without GDC]***

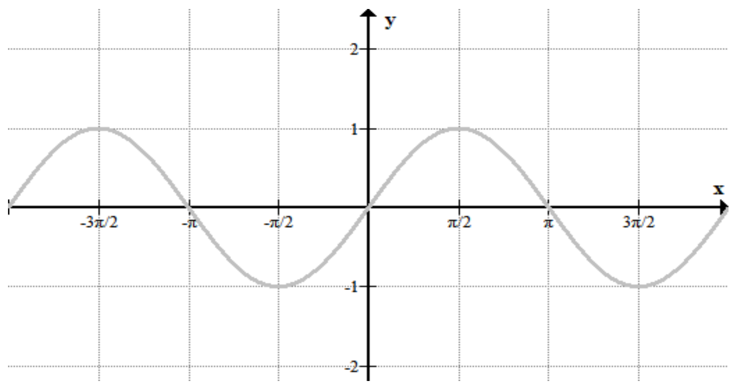
The following diagrams show part of the graph of $y = \sin x$.

(a) On the diagram below sketch the graph of the function $f(x) = \sin 2x$.



[2]

(b) On the diagram below sketch the graph of the function $f(x) = \sin \frac{x}{2}$.



[2]

(c) Complete the following table.

Function	Amplitude	Period	Central axis	Range
$y = \sin x$	1	2π	$y = 0$	$-1 \le y \le 1$
$f(x) = \sin 2x$				
$g(x) = \sin \frac{x}{2}$				

[4]

(d) Write down the **period** for the function $f(x) = \sin bx$. [1]

(e) For the function $f(x) = a \sin bx + c$, write down

(i) the amplitude (ii) the equation of the central axis. (iii) the period. [3]

4. [Maximum mark: 8] ***[without GDC]***

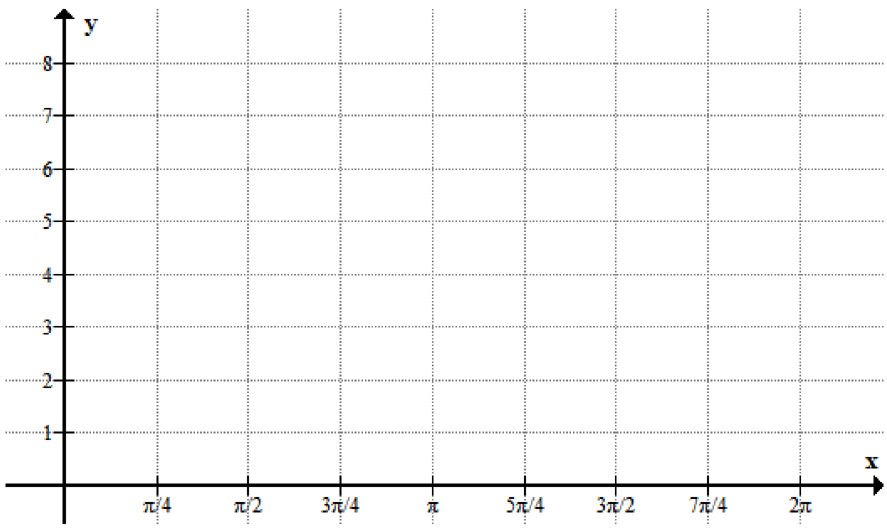
Let $f(x) = 3\sin 2x + 4$, $0 \le x \le 2\pi$

(a) Complete the table

Function	Amplitude	Period	Central axis	Range
$f(x)$				

[4]

(b) On the diagram below, draw the graph of the function f.



[4]

5. [Maximum mark: 8] ***[without GDC]***

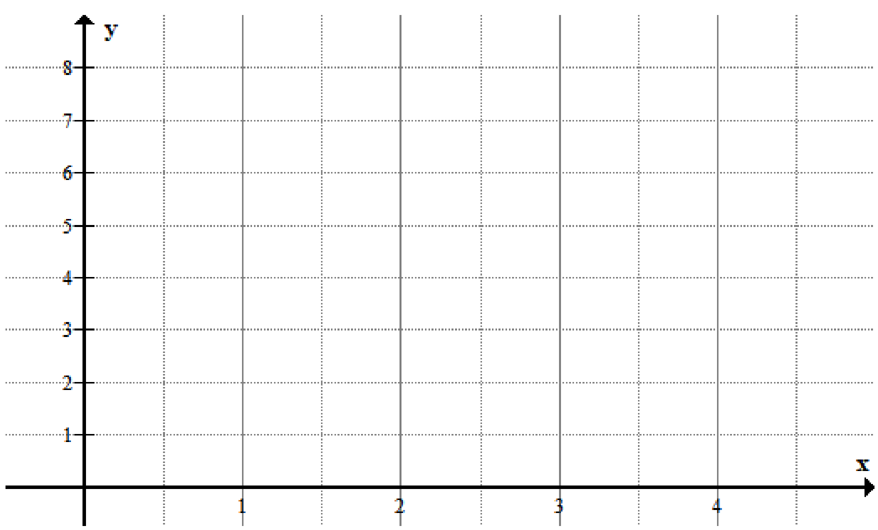
Let $f(x) = 3\sin \pi x + 4$, $0 \le x \le 4$

(a) Complete the table

Function	Amplitude	Period	Central axis	Range
$f(x)$				

[4]

(b) On the diagram below, draw the graph of the function f.



[4]

6. [Maximum mark: 7] ***[without GDC]***

(a) For the function $f(x) = a\cos bx + c$, where $a > 0$, complete the table below.

Function	Amplitude	Period	Central axis	Range
$f(x)$				

[4]

(b) For the function $f(x) = a\tan bx + c$, complete the table below.

Function	Period	Central axis	Range
$f(x)$			

[3]

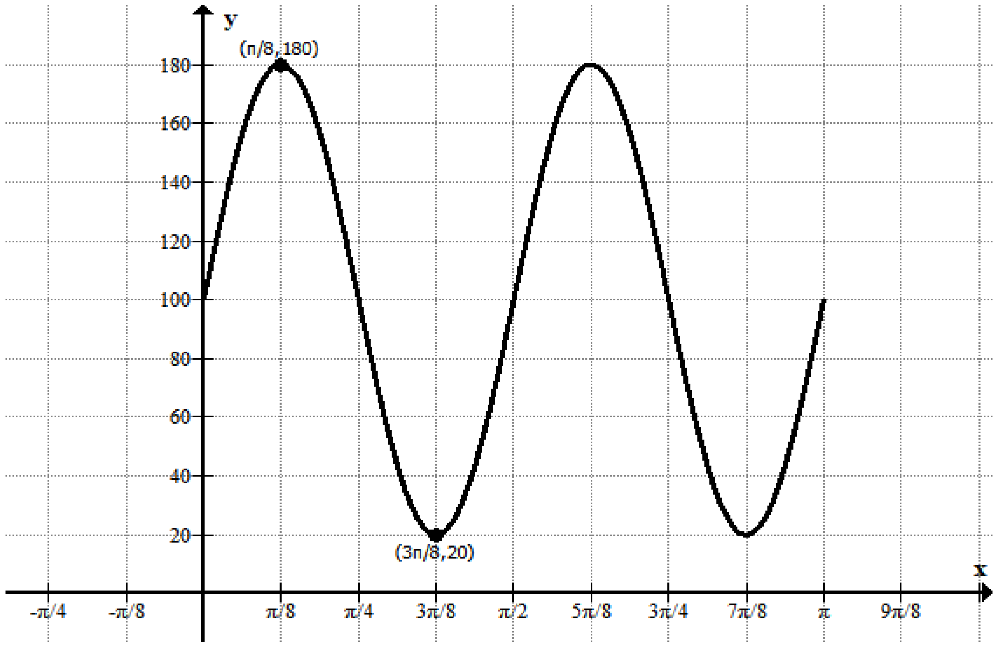
7. [Maximum mark: 14] ***[without GDC]***

Complete the following table

Function	Amplitude	Period	Central axis	Range
$f(x) = \sin x$	1	2π	$y = 0$	$-1 \le y \le 1$
$f(x) = \cos x$				
$f(x) = \sin x + 1$				
$f(x) = \sin x - 1$				
$f(x) = 5\sin x$				
$f(x) = -7\sin x$				
$f(x) = \sin 4x$				
$f(x) = -\cos 4x$				
$f(x) = 3\sin 4x$				
$f(x) = 3\sin 4x + 10$				
$f(x) = 3\sin 4x - 2$				
$f(x) = -5\sin 3x$				
$f(x) = -5\sin x + 10$				
$f(x) = \tan x$		π	$y = 0$	$y \in R$
$f(x) = \tan 4x$				
$f(x) = 5\tan 4x + 10$				

8*. [Maximum mark: 12] ***[without GDC]***

Part of the graph of a trigonometric function $f(x)$ is given below. There is a local maximum at $(\pi/8, 180)$ and a local minimum at $(3\pi/8, 20)$.



(a) Write down the values of

(i) the amplitude (ii) the central value (iii) the period [3]

(b) Express the function in the form $f(x) = A\sin Bx + C$ [3]

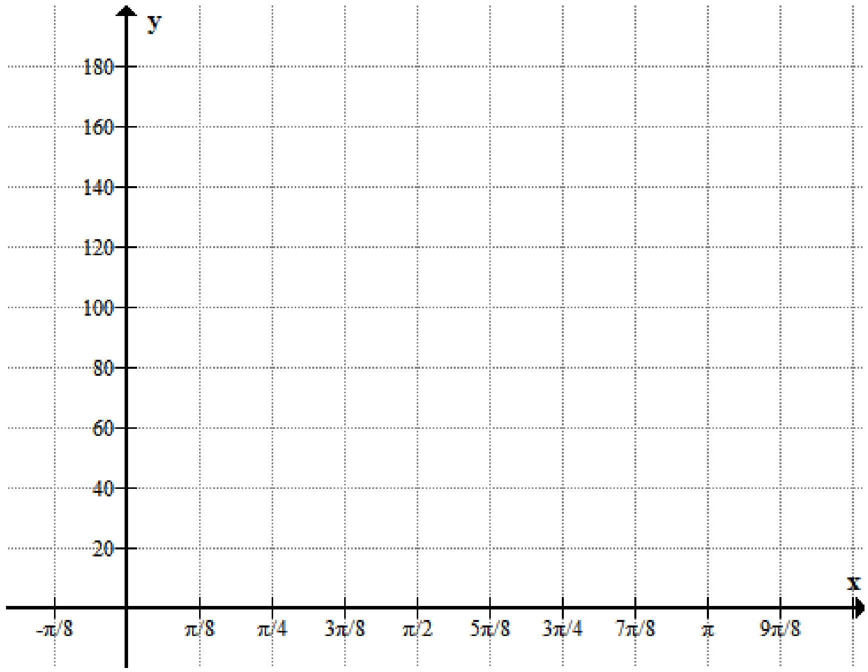
(c) Complete the following table by expressing $f(x)$ in three alternative forms. [6]

$f(x) = -80\sin[B(x-D)] + C$	
$f(x) = 80\cos[B(x-D)] + C$	
$f(x) = -80\cos[B(x-D)] + C$	

9. [Maximum mark: 5] ***[without GDC]***

Sketch the graph of the function

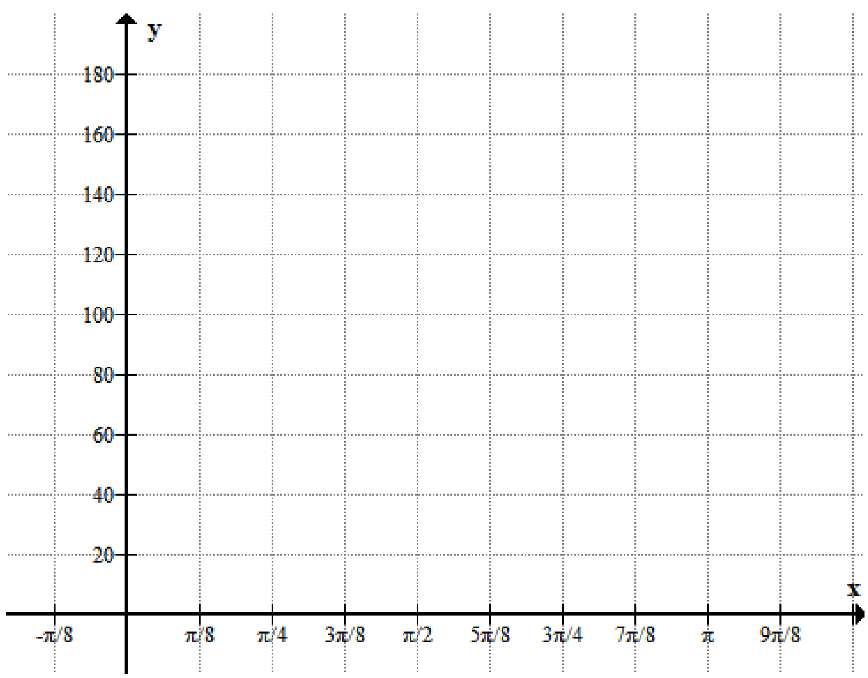
$$f(x) = 60\sin 4x + 100\,,\quad 0 \le x \le \pi$$



10. [Maximum mark: 5] ***[without GDC]***

Sketch the graph of the function

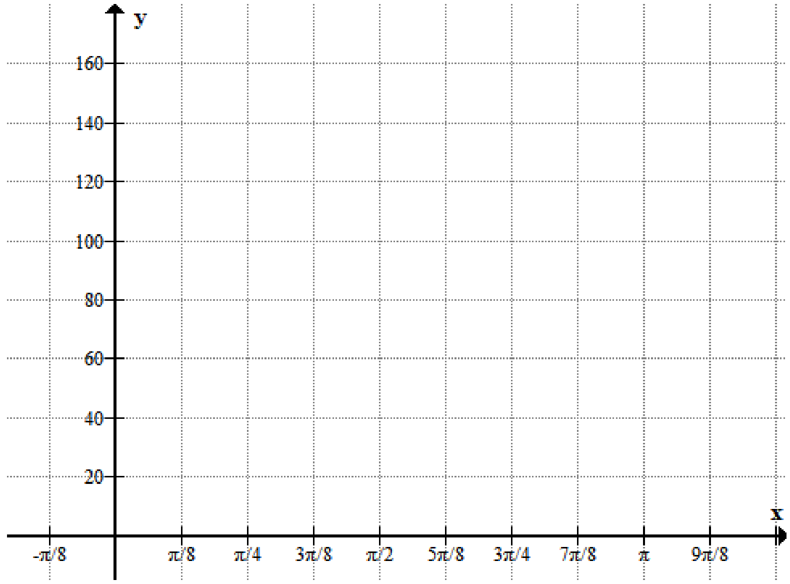
$$f(x) = 60\cos 4x + 100\,,\quad 0 \le x \le \pi$$



11. [Maximum mark: 5] ***[without GDC]***

Sketch the graph of the function

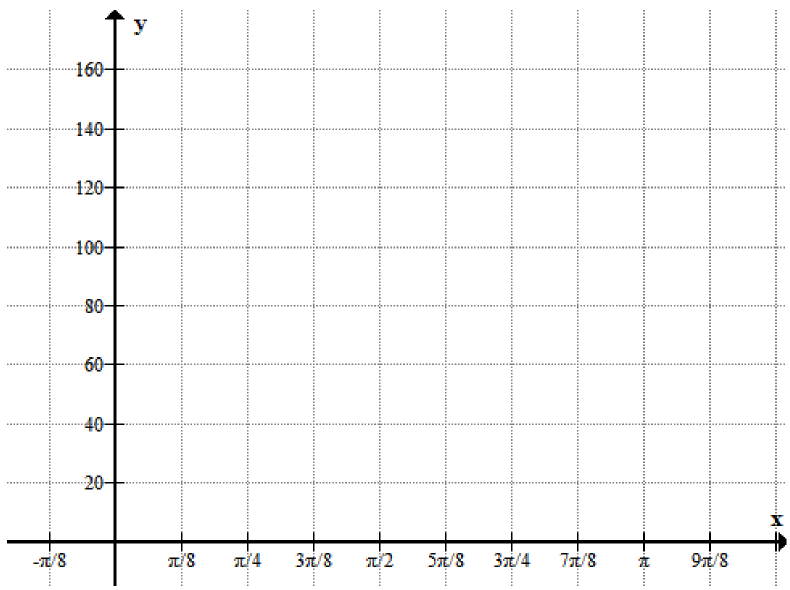
$$f(x) = -60\sin 4x + 80\,,\quad 0 \le x \le \pi$$



12. [Maximum mark: 9] ***[without GDC]***

(a) Sketch the graph of the function

$$f(x) = -60\cos 4x + 80\,,\quad 0 \le x \le \pi$$



[5]

(b) Write down the possible values of k if the equation $f(x) = k$ has

(i) exactly two solution ....

(ii) exactly three solutions ....

(iii) exactly four solutions ....

(iv) no solutions .... [4]

A. Exam style questions (SHORT)

13. [Maximum mark: 7] ***[without GDC]***

The function f is defined by $f(x) = 30\sin 3x \cos 3x$, $0 \le x \le \frac{\pi}{3}$

(a) Write down an expression for $f(x)$ in the form $a\sin 6x$, where a is an integer. [2]

(b) Find the period of f . [2]

(c) Solve $f(x) = 0$, giving your answers in terms of π. [3]

14. [Maximum mark: 4] ***[with / without GDC]***

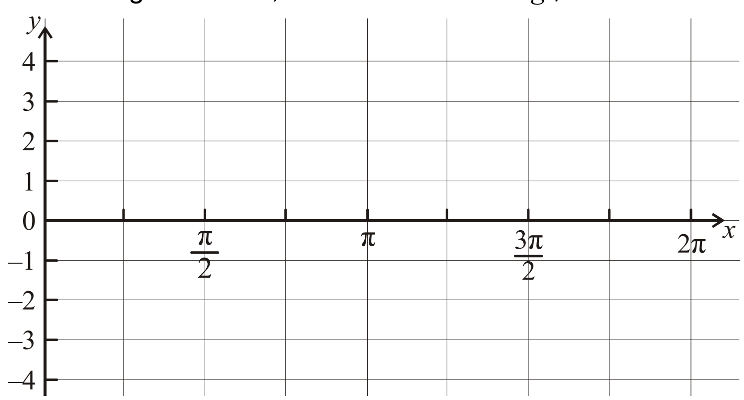
Let $f(x) = 4\sin\left(3x + \frac{\pi}{2}\right)$. For what values of k will $f(x) = k$ have no solutions?

15. [Maximum mark: 6] ***[without GDC]***

Consider $g(x) = 3\sin 2x$.

(a) Write down the period of g. [1]

(b) On the diagram below, sketch the curve of g, for $0 \le x \le 2\pi$.

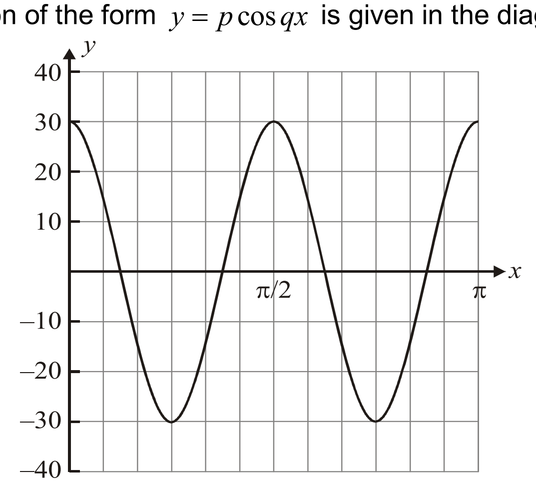


[3]

(c) Write down the number of solutions to the equation $g(x) = 2$, for $0 \le x \le 2\pi$. [2]

16. [Maximum mark: 4] ***[without GDC]***

The graph of a function of the form $y = p\cos qx$ is given in the diagram below.



(a) Write down the value of p. [2]

(b) Calculate the value of q. [2]

17. [Maximum mark: 6] ***[with / without GDC]***

Let $f(x)=\sin 2x$ and $g(x)=\sin(0.5x)$.

(a) Write down

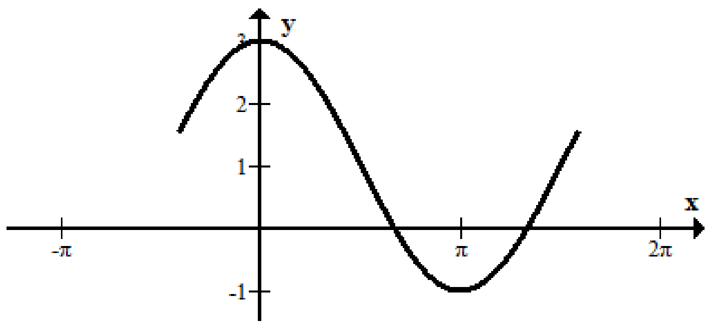
(i) the minimum value of the function f

(ii) the period of the function g. [3]

(b) Consider the equation $f(x)=g(x)$. Find the number of solutions, for $0 \le x \le \frac{3\pi}{2}$ [3]

18. [Maximum mark: 4] ***[without GDC]***

Part of the graph of $y=p+q\cos x$ is shown below. The graph passes through the points $(0, 3)$ and $(\pi, -1)$.



Find the value of (i) p; (ii) q.

19. [Maximum mark: 6] ***[without GDC]***

Let $f(x)=\frac{3x}{2}+1$, $g(x)=4\cos\left(\frac{x}{3}\right)-1$. Let $h(x)=(g\circ f)(x)$.

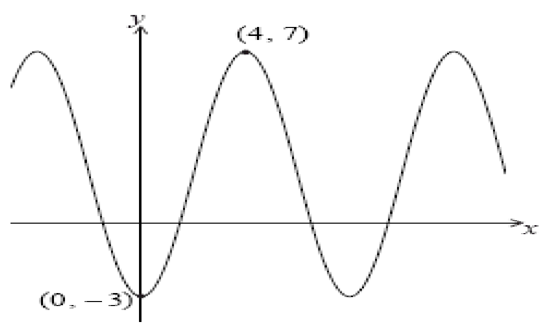
(a) Find an expression for $h(x)$ [3]

(b) Write down the period of h. [1]

(c) Write down the range of h. [2]

20. [Maximum mark: 7] ***[without GDC]***

The graph of $y=p\cos qx+r$, for $-5 \le x \le 14$, is shown below.



There is a minimum point at (0, –3) and a maximum point at (4, 7).

(a) Find the value of (i) p; (ii) q; (iii) r. [6]

(b) The equation $y=k$ has exactly **two** solutions. Write down the value of k. [1]

21. [Maximum mark: 4] ***[without GDC]***

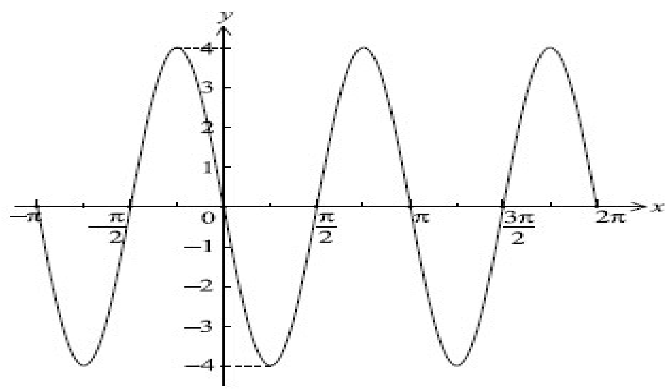
The depth, y metres, of sea water in a bay t hours after midnight may be represented by the function $y = a + b\cos\left(\frac{2\pi}{k}t\right)$ where a, b and k are constants.

The water is at a maximum depth of 14.3 m at midnight and noon, and is at a minimum depth of 10.3 m at 06:00 and at 18:00.

Write down the value of (i) a; (ii) b; (iii) k.

22*. [Maximum mark: 6] ***[without GDC]***

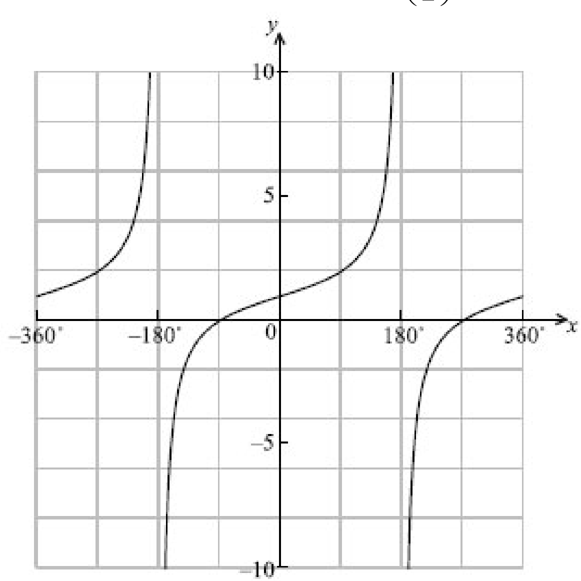
Let $f(x) = a\sin b(x-c)$. Part of the graph of f is given below.



Given that a, b and c are positive, find the value of a, of b and of c.

23. [Maximum mark: 6] ***[without GDC]***

The diagram below shows the graph of $f(x) = 1 + \tan\left(\frac{x}{2}\right)$, for $-360° \le x \le 360°$.



(a) On the same diagram, draw the asymptotes. [2]

(b) Write down
  (i) the period of the function;
  (ii) the value of $f(90°)$. [2]

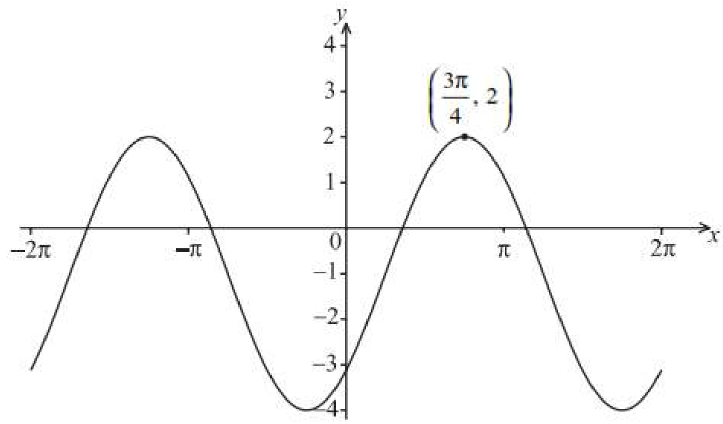
(c) Solve $f(x) = 0$ for $-360° \le x \le 360°$. [2]

24. [Maximum mark: 6] ***[without GDC]***

The graph of $y = \cos x$ is transformed into the graph of $y = 8 - 2\cos\frac{\pi x}{6}$. Find a sequence of simple geometric transformations that does this.

25*. [Maximum mark: 6] ***[without GDC]***

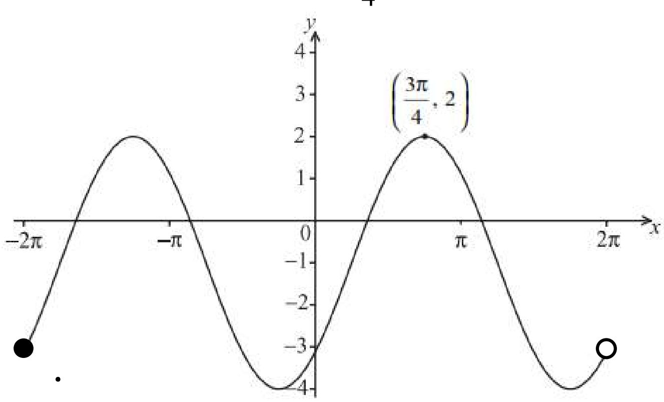
The graph below represents $f(x) = a\sin(x + b) + c$, where $a, b,$ and c are constants.



Find values for $a, b,$ and c.

26. [Maximum mark: 6] ***[without GDC]***

The graph below represents $f(x) = 3\sin(x - \frac{\pi}{4}) - 1$, $-2\pi \le x < 2\pi$



(a) Write down the range of the equation f. [2]

(b) Write down the number of solutions of the equation $f(x) = -2$. [1]

(c) Write down the values of k, for which the equation $f(x) = k$, for $-2\pi \le x < 2\pi$,

(i) has exactly 2 solutions; (ii) has exactly 4 solutions; (iii) has no solutions. [3]

27*. [Maximum mark: 6] ***[with / without GDC]***

The depth, $h(t)$ metres, of water at the entrance to a harbor at t hours after midnight on a particular day is given by $h(t) = 8 + 4\sin\left(\frac{\pi t}{6}\right)$, $0 \le t \le 24$

(a) Find the maximum depth and the minimum depth of the water . [2]

(b) Find the values of t for which $h(t) \ge 8$. [4]

28. [Maximum mark: 4/8] ***[with / without GDC]***

Consider $y = \sin\left(x + \frac{\pi}{9}\right)$.

(a) The graph of y intersects the x-axis at point A. Find the x-coordinate of A, where $0 \le x \le \pi$. [2/4]

(b) Solve the equation $\sin\left(x + \frac{\pi}{9}\right) = -\frac{1}{2}$, for $0 \le x \le 2\pi$. [2/4]

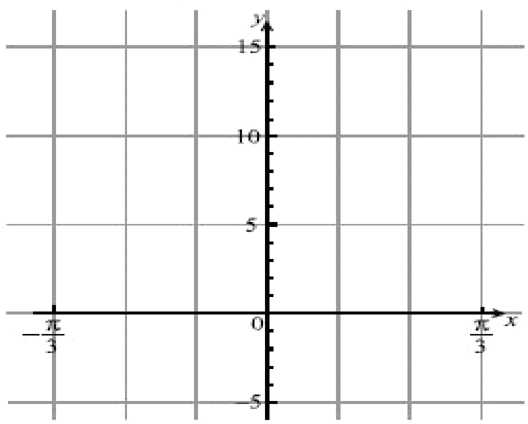
METHOD A [with GDC] *[4 marks]*

(**) METHOD B [without GDC] *[8 marks]*

29. [Maximum mark: 5] ***[with GDC]***

Let $f(x) = 4\tan^2 x - 4\sin x$, $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$.

(a) On the grid below, sketch the graph of $y = f(x)$.

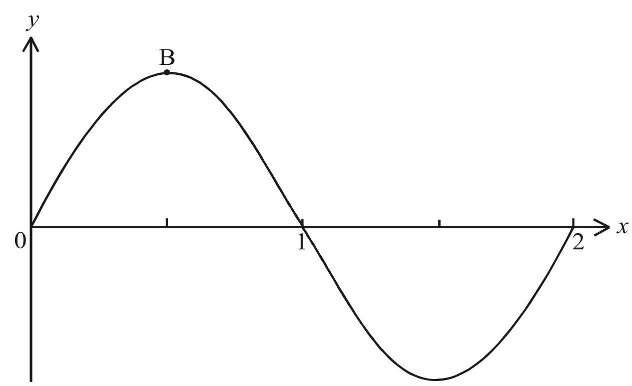


[3]

(b) Solve the equation $f(x) = 1$. [2]

30. [Maximum mark: 6] ***[with GDC]***

Let $f(x) = 6\sin \pi x$ and $g(x) = 6e^{-x} - 3$, for $0 \le x \le 2$. The graph of f is shown on the diagram below. There is a maximum value at B $(0.5, b)$.



(a) Write down the value of b. [1]

(b) On the same diagram, sketch the graph of g. [3]

(c) Solve $f(x) = g(x)$, $0.5 \le x \le 1.5$. [2]

31. [Maximum mark: 6] ***[with GDC]***

Let $f(x) = 2\sin(3x-1)$ and $g(x) = x^2$.

(a) Describe the sequence of transformations from $y = \sin x$ to $f(x)$. [4]

(b) Solve the inequality $f(x) \geq g(x)$. [2]

32. [Maximum mark: 5] ***[with GDC]***

By observing the graph of the function $f(x) = \sin 3x + \sin 6x$, $0 < x < 2\pi$

(a) Write down the range of the function f. [2]

(b) Write down the number of solutions of the equation $f(x) = 0$. [1]

(c) Write down the exact period of the function f. [2]

33.** [Maximum mark: 6] ***[with GDC]***

Consider the functions $f(x) = e^{2x}$ and $g(x) = \sin\dfrac{\pi x}{2}$ (x in radians).

(a) Find the period of the function $f \circ g$. [3]

(b) Find the intervals for which $(f \circ g)(x) > 4$. [3]

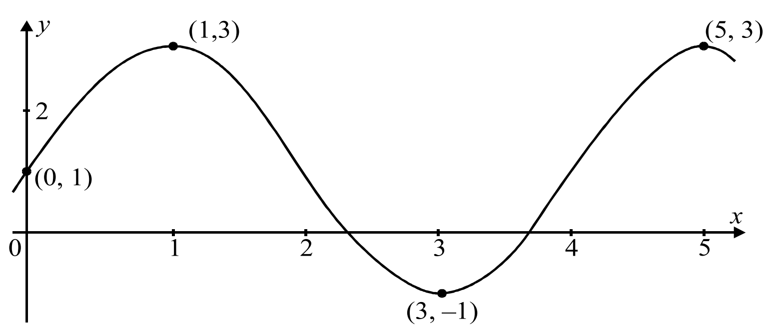
B. Exam style questions (LONG)

34. [Maximum mark: 18] ***[without GDC]***

The diagram shows the graph of the function f given by

$$f(x) = A\sin\left(\frac{\pi}{2}x\right) + B, \quad \text{for } 0 \leq x \leq 5,$$

where A and B are constants, and x is measured in radians.



The graph includes the points (1, 3) and (5, 3), which are maximum points of the graph.

(a) Write down the values of $f(1)$ and $f(5)$. [2]

(b) Show that the period of f is 4. [2]

The point (3, –1) is a minimum point of the graph.

(c) Show that $A = 2$, and find the value of B. [5]

(d) Solve the equation $f(x) = 2$ for $0 \le x \le 5$. [5]

(e) Consider the equation $f(x) = k$, for $0 \le x \le 5$.

Write down the possible values of k if the equation has

(i) exactly one solution (ii) exactly three solutions

(iii) exactly two solutions (iv) no solutions [4]

35. [Maximum mark: 13] ***[with GDC]***

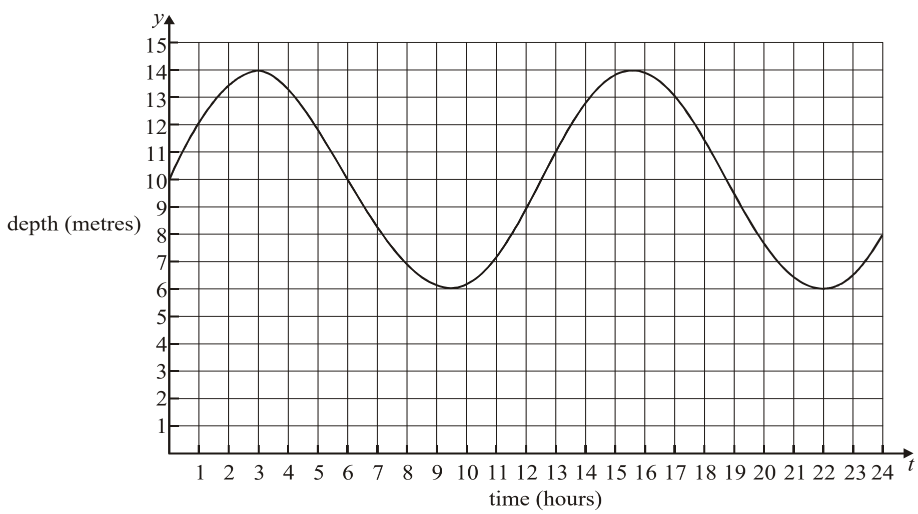
The depth y metres of water in a harbour is given by the equation

$$y = 10 + 4\sin\left(\frac{t}{2}\right).$$

where t is the number of hours after midnight.

(a) Calculate the depth of the water (i) when $t = 2$; (ii) at $21{:}00$. [3]

The sketch below shows the depth y, of water, at time t, during one day (24 hours).



(b) (i) Write down the maximum depth of water in the harbour.

(ii) Calculate the value of t when the water is first at its maximum depth during the day. [3]

The harbour gates are closed when the depth of the water is less than seven metres.

An alarm rings when the gates are opened or closed.

(c) (i) How many times does the alarm sound during the day?

(ii) Find the value of t when the alarm sounds first.

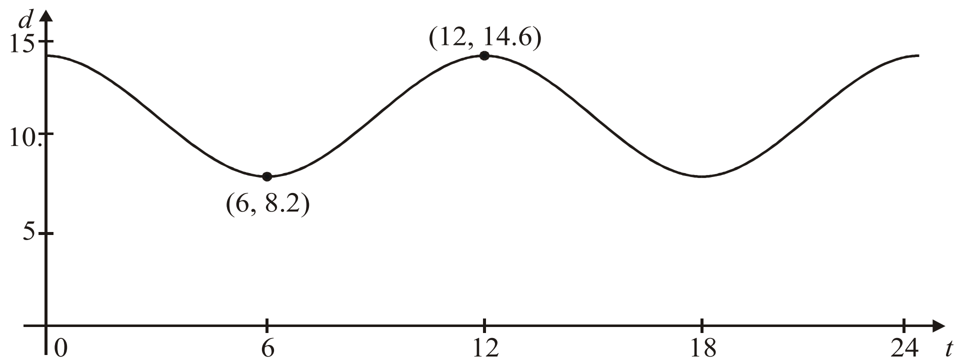
(iii) Use the graph to find the length of time during the day when the harbour gates are closed. Give your answer in hours, to the nearest hour. [7]

36. [Maximum mark: 10] ***[with GDC]***

A formula for the depth d metres of water in a harbour at a time t hours after midnight is

$$d = P + Q\cos\left(\frac{\pi}{6}t\right),\quad 0 \le t \le 24,$$

where P and Q are positive constants. In the following graph the point (6, 8.2) is a minimum point and the point (12, 14.6) is a maximum point.



(a) Find the value of (i) Q; (ii) P. [3]

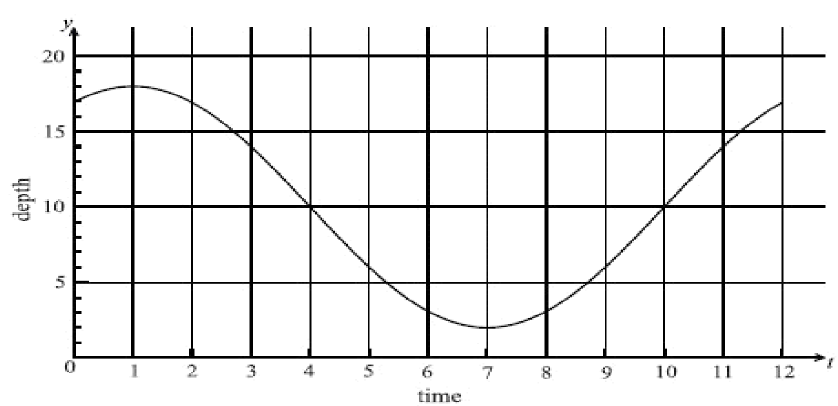
(b) Find the **first** time in the 24-hour period when the depth of the water is 10 metres. [3]

(c) (i) Use symmetry of the graph to find the **next** time when the depth d is 10m.

(ii) Hence find the time intervals in the 24-hour period during which the water is less than 10 metres deep. [4]

37. [Maximum mark: 11] ***[with GDC]***

The following graph shows the depth of water, y metres, at a point P, during one day. The time t is given in hours, from midnight to noon.



(a) Use the graph to write down an estimate of the value of t when

(i) the depth of water is minimum;

(ii) the depth of water is maximum;

(iii) the depth of the water is increasing most rapidly. [3]

(b) The depth of water can be modelled by the function $y = A\cos(B(t-1)) + C$.

(i) Show that $A = 8$.

(ii) Write down the value of C.

(iii) Find the value of B. [6]

(c) A sailor knows that he cannot sail past P when the depth of the water is less than 12 m. Calculate the values of t between which he cannot sail past P. [2]

38. [Maximum mark: 13] ***[with GDC]***

Let $f(x) = 3\sin x + 4\cos x$, for $-2\pi \le x \le 2\pi$.

(a) Sketch the graph of f. [3]

(b) Write down

(i) the amplitude; (ii) the period; (iii) the x-intercept between $-\pi/2$ and 0. [3]

(c) Hence write $f(x)$ in the form $p\sin(qx + r)$. [3]

(d) Write down the x-coordinates of the points where f has a maximum. [2]

(e) Write down the two values of k for which the equation $f(x) = k$ has exactly two solutions. [2]

39. [Maximum mark: 10] ***[with GDC]***

A spring is suspended from the ceiling. It is pulled down and released, and then oscillates up and down. Its length, l centimetres, is modelled by the function

$$l = 33 + 5\cos\left((720t)^\circ\right),$$

where t is time in seconds after release.

(a) Find the length of the spring after 1 second. [2]

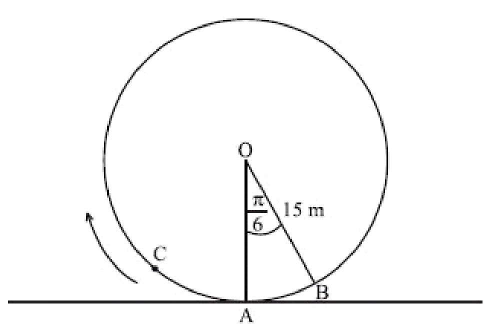
(b) Find the minimum length of the spring. [3]

(c) Find the first time at which the length is 33 cm. [3]

(d) What is the period of the motion? [2]

40*. [Maximum mark: 20] ***[with / without GDC]***

A Ferris wheel with centre O and a radius of 15 metres is represented in the diagram below. Initially seat A is at ground level. The next seat is B, where $A\hat{O}B = \dfrac{\pi}{6}$.



(a) Find the length of the arc AB. [2]

(b) Find the area of the sector AOB. [2]

(c) The wheel turns clockwise through $\dfrac{2\pi}{3}$. Find the height of A above the ground. [3]

The height, h metres, of seat A above the ground after t minutes, can be modelled by the function $h(t) = -a\cos 2t + c$.

(d) (i) Find the values of a and c.

(ii) Find the time for a complete turn of the wheel. [5]

The height, h metres, of seat C above the ground after t minutes, can be modelled by the function

$$h(t) = 15 - 15\cos\left(2t + \frac{\pi}{4}\right)$$

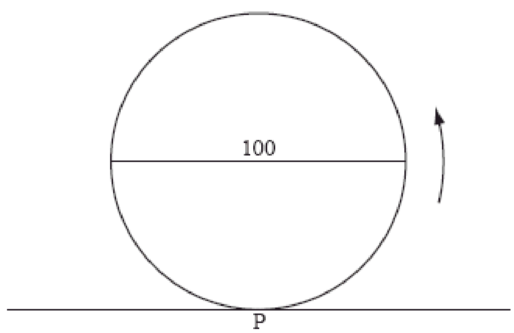
(e) (i) Find the height of seat C when $t = \frac{\pi}{4}$.

(ii) Find the initial height of seat C.

(iii) Find the time at which seat C first reaches its highest point. [8]

41*. [Maximum mark: 14] ***[without GDC]***

The following diagram represents a large Ferris wheel, with a diameter of 100 metres.



Let P be a point on the wheel. The wheel starts with P at the lowest point, at ground level. The wheel rotates at a constant rate, in an anticlockwise (counterclockwise) direction. One revolution takes 20 minutes.

(a) Write down the height of P above ground level after (i) 10 min; (ii) 15 min.

Let $h(t)$ metres be the height of P above ground level after t minutes. Some values of $h(t)$ are given in the table below.

t	$h(t)$
0	0.0
1	2.4
2	9.5

t	$h(t)$
3	20.6
4	34.5
5	50.0

(b) (i) Show that $h(8) = 90.5$. (ii) Find $h(21)$. [4]

(c) **Sketch** the graph of h, for $0 \le t \le 40$. [3]

(d) Given that h can be expressed in the form $h(t) = a\cos bt + c$, find a, b and c. [5]

42. [Maximum mark: 12] ***[with GDC]***

Let $f(x)=5\cos\frac{\pi}{4}x$ and $g(x)=-0.5x^2+5x-8$, for $0\le x\le 9$.

(a) On the same diagram, sketch the graphs of f and g. [3]

(b) Consider the graph of f. Write down

(i) the x-intercept that lies between $x=0$ and $x=3$;

(ii) the period;

(iii) the amplitude. [4]

(c) Consider the graph of g. Write down

(i) the two x-intercepts;

(ii) the equation of the axis of symmetry. [3]

(d) Find the x-coordinates of the points of intersection between f and g. [2]

43. [Maximum mark: 11] ***[with GDC]***

(a) Consider the equation $x^2+kx+1=0$. For what values of k does this equation have two **equal** roots? [3]

Let f be the function $f(\theta)=2\cos 2\theta+4\cos\theta+3$, for $-360°\le x\le 360°$.

(b) Show that this function may be written as $f(\theta)=4\cos^2\theta+4\cos\theta+1$. [1]

(c) Consider the equation $f(\theta)=0$, for $-360°\le x\le 360°$.

(i) How many distinct values of $\cos\theta$ satisfy this equation?

(ii) Find all values of θ which satisfy this equation. [5]

(d) Given that $f(\theta)=c$ is satisfied by only three values of θ, find the value of c. [2]