*INTERNATIONAL BACCALAUREATE*

***Mathematics: analysis and approaches***

# *Math AA*

***EXERCISES [Math-AA 3.1-3.3]***

***3D GEOMETRY – TRIANGLES***

*Compiled by Christos Nikolaidis*

## O. Practice questions

### 3D GEOMETRY

**1.** [Maximum mark: 7] ***[without GDC]***

Let A(2,-3,5) and B(-1,1,5). Find

(a) the distance between A and B. [2]

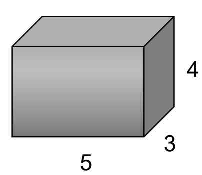
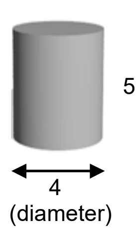
(b) the distance between O and B. [1]

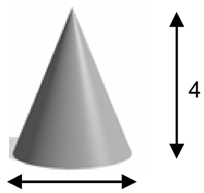
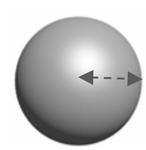
(c) the coordinates of the midpoint M of the line segment [AB]. [2]

(d) the coordinates of point C given that B is the midpoint of [AC]. [2]

**2.** [Maximum mark: 16] ***[without GDC]***

Complete the table

| Solid | Volume | Surface area |
|---|---|---|
| cuboid   | | |
| cylinder   | | |

| | | |
|---|---|---|
| cone   6 (diameter) | | |
| sphere  radius = 3 | | |

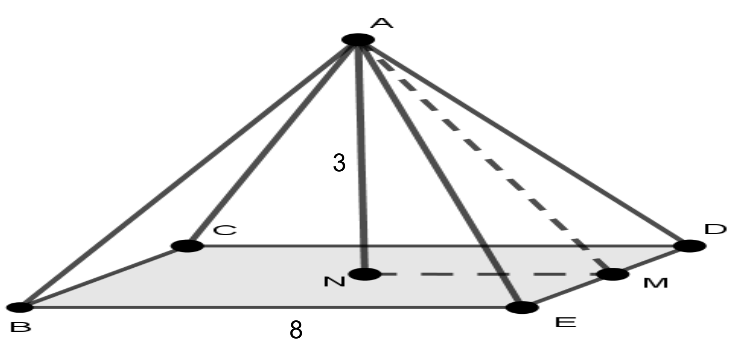
for each shape [1+3]

**3.** [Maximum mark: 7] ***[without GDC]***

For a right pyramid of square base of side 8 and vertical height 3 find

(a) the volume [2]

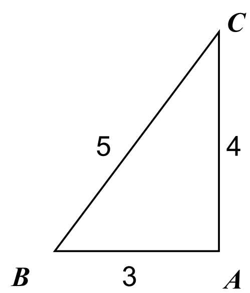
(b) the surface area [5]



## TRIANGLES

**4.** [Maximum mark: 14] ***[without GDC]***

Consider the following right-angled triangle, where $\hat{A}$ = 90°




(a) Complete the tables

| | |
|---|---|
| $\sin\hat{B}$ | |
| $\cos\hat{B}$ | |
| $\tan\hat{B}$ | |

| | |
|---|---|
| $\sin\hat{C}$ | |
| $\cos\hat{C}$ | |
| $\tan\hat{C}$ | |

[6]

(b) Confirm that the **sine rule** holds. (It is known that $\sin\hat{A}=1$)

| | | |
|---|---|---|
| $\dfrac{a}{\sin\hat{A}}=\dfrac{5}{1}=5$ | $\dfrac{b}{\sin\hat{B}}=$ | $\dfrac{c}{\sin\hat{C}}=$ |

[2]

(c) Confirm that all three versions of the **cosine rule** hold.
(the first version is given below; it is known that $\cos\hat{A}=0$)

| LHS | RHS |
|---|---|
| $5^2$ | $3^2+4^2-2(3)(4)\cos\hat{A}=9+16-0=25$ |
| $3^2$ | |
| $4^2$ | |

[4]

(d) Find the area of the triangle, by using all the three versions of the formula

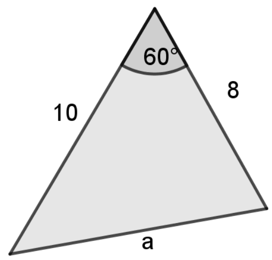
$Area=\dfrac{1}{2}ab\sin\hat{C}$ (the first version is given below)

| |
|---|
| $Area=\dfrac{1}{2}\times3\times4\times\sin\hat{A}=6$ |
| $Area=$ |
| $Area=$ |

[2]

**5.** [Maximum mark: 4] ***[with / without GDC]***

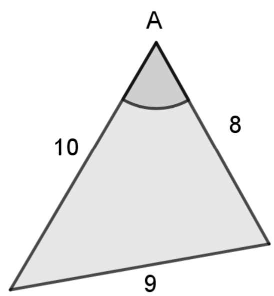
Use the **cosine rule** to find the size of side $a$.




**6.** [Maximum mark: 5] ***[with GDC]***

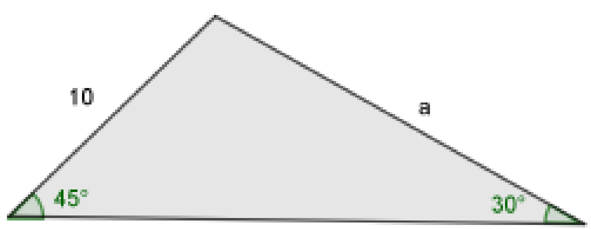
(a) Use the **cosine rule** to find the cosine of angle A. [4]

(b) Hence find the size of angle A. [1]




**7.** [Maximum mark: 4] ***[with / without GDC]***

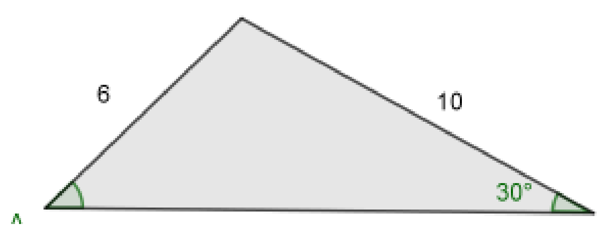
Use the **sine rule** to find the size of side $a$.




**8.** [Maximum mark: 6] ***[with GDC]***

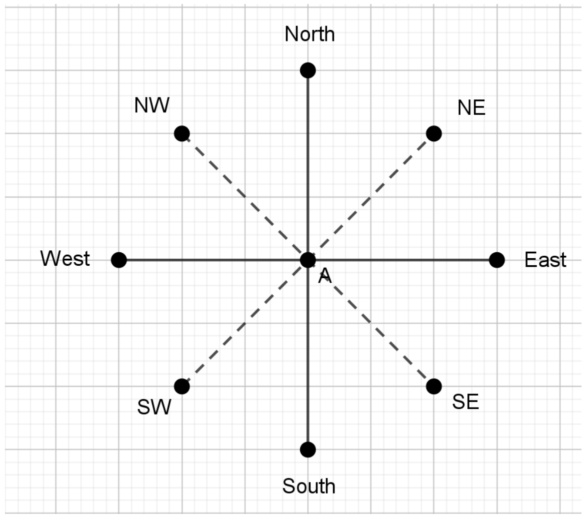
(a) Use **the sine rule** to find the sine of angle A. [4]

(b) Hence find the **two possible** values of angle A. [2]



**9.** [Maximum mark: 10] ***[with GDC]***

Point A is at the center of the following diagram.




Bill and Chris and Dianna are located at point A and start moving,

Bill to the **NE** at point **B**

Chris to the **South** at point **C**

Dianna to the **West** at point **D**

(a) Write down the size of angle BAC. [1]

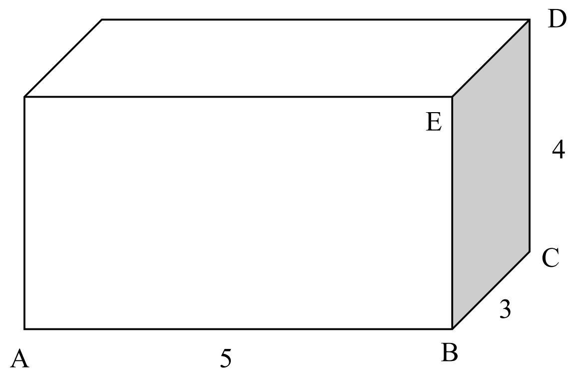
(b) Write down the bearing of the course of each person. [3]

(c) Find the bearing of the course from B to A. [2]

(d) Given that AB = 2 km and AC is 3km, Find the distance between B and C. [3]

**10.** [Maximum mark: 14] ***[with GDC]***

Consider the following cuboid of dimensions $5 \times 3 \times 4$, as shown.




(a) Find the length AC. [3]

(b) Find the length AD. [3]

(c) Find the angle of elevation from A to E. [3]

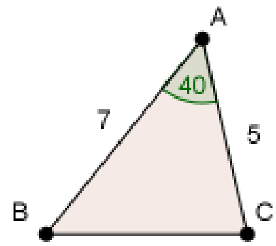
(d) Find the angle of elevation from A to D. [3]

(e) Find the angle of depression from E to A. [2]

**11*.** [Maximum mark: 30] ***[with GDC]***

In each of the following triangles one of the angles has size $40^{\circ}$, two of the sides have lengths 5 and 7 respectively.

(a) For the following triangle

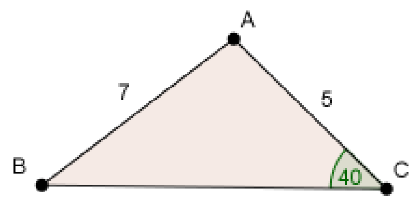



(i) Find the area of the triangle

(ii) Find BC

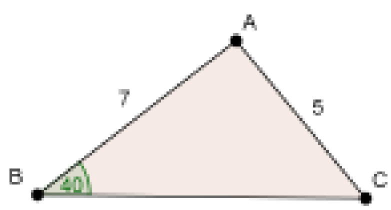
(iii) Find the size of $\hat{B}$ and **hence** the size of $\hat{C}$. [7]

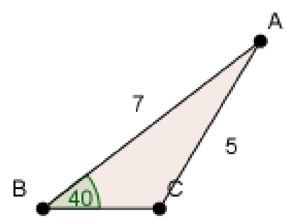
(b) For the following triangle




find the size of $\hat{B}$ and **hence** the size of $\hat{A}$. [5]

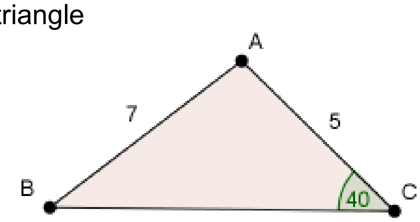
(c) For each of the following triangles (ambiguous case)




find the size of $\hat{C}$ and **hence** the size of $\hat{A}$. [6]

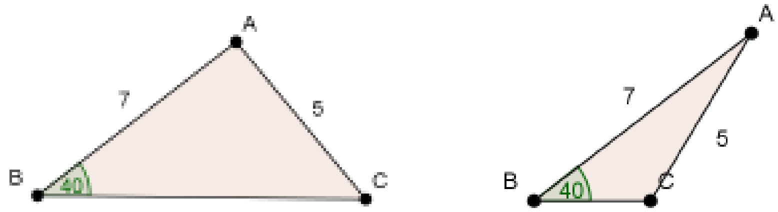
(d) For the following triangle




(i) Use the cosine rule to directly find the side BC.

(ii) **Hence** find the area of the triangle. [5]

(e) For the following triangles (ambiguous case)




(i) Use the cosine rule to directly find the side BC of each triangle.

(ii) **Hence** find the area of each triangle. [7]

## A. Exam style questions (SHORT)

**12.** [Maximum mark: 6] ***[with GDC]***

The following diagram shows triangle ABC.

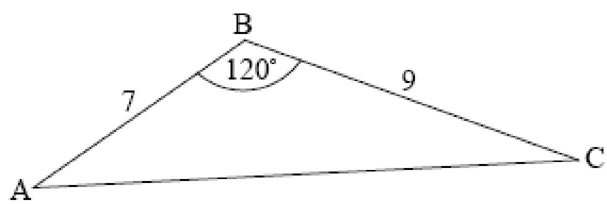



***diagram not to scale***

AB = 7 cm, BC = 9 cm and $A\hat{B}C$ = 120°.

(a) Find AC. [3]

(b) Find $B\hat{A}C$. [3]

**13.** [Maximum mark: 4] ***[with GDC]***

A triangle has sides of length 4, 5, 7 units. Find, to the nearest tenth of a degree, the size of the largest angle.

**14.** [Maximum mark: 4] ***[with GDC]***

The following diagram shows a triangle with sides 5 cm, 7 cm, 8 cm.

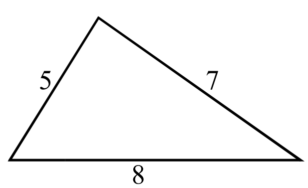



**Diagram not to scale**

(a) Find the size of the smallest angle, in degrees; [2]

(b) Find the area of the triangle. [2]

**15.** [Maximum mark: 6] ***[with GDC]***

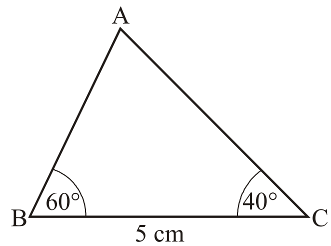
In the triangle PQR, PR = 5 cm, QR = 4 cm and PQ = 6 cm. Calculate

(a) the size of $\mathrm{P\hat{Q}R}$ ; [4]

(b) the area of triangle PQR. [2]

**16.** [Maximum mark: 6] ***[with GDC]***

The following diagram shows a triangle ABC, where BC = 5 cm, $\hat{B}$ = 60°, $\hat{C}$ = 40°.

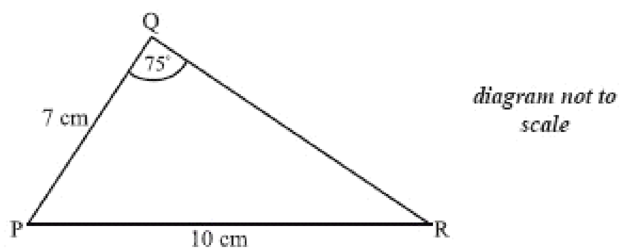



(a) Calculate AB. [3]

(b) Find the area of the triangle. [3]

**17.** [Maximum mark: 6] ***[with GDC]***

The diagram below shows triangle PQR. The length of [PQ] is 7 cm, the length of [PR] is 10 cm, and $\mathrm{P\hat{Q}R}$ is 75°.

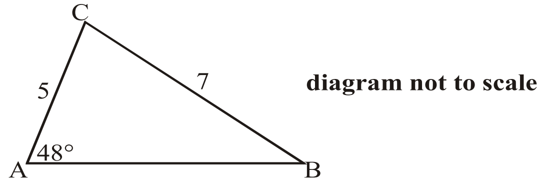



(a) Find $\mathrm{P\hat{R}Q}$ [3]

(b) Find the area of triangle PQR. [3]

**18.** [Maximum mark: 6] ***[with GDC]***

In triangle ABC, AC = 5, BC = 7, $\hat{A}$ = 48°, as shown in the diagram.

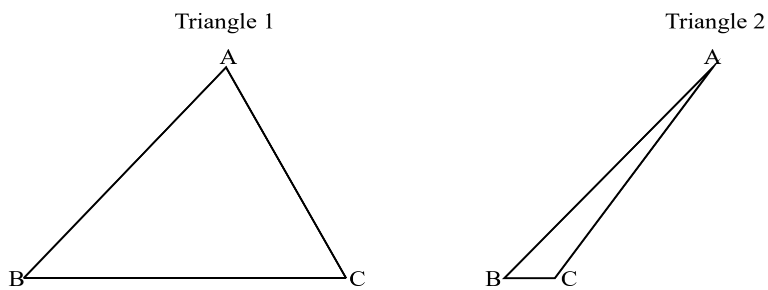



Find $\hat{B}$, giving your answer correct to the nearest degree.

**19*.** [Maximum mark: 4] ***[with GDC]***

The diagrams below show two triangles both satisfying the conditions

AB = 20 cm, AC = 17 cm, $A\hat{B}C$ = 50°.




**Diagrams not to scale**

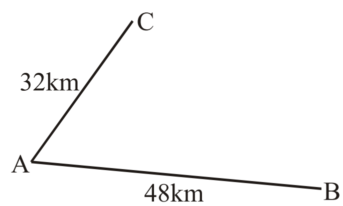
Calculate

(a) the size of $A\hat{C}B$ in **Triangle 2.** [2]

(b) the area of **Triangle 1**. [2]

**20.** [Maximum mark: 4] ***[with GDC]***

Town A is 48 km from town B and 32 km from town C as shown in the diagram.




Given that town B is 56 km from town C, find the size of angle $C\hat{A}B$ to the nearest degree.

**21*.** [Maximum mark: 6] ***[with GDC]***

Two boats A and B start moving from the same point P. Boat A moves in a straight line at 20 km $h^{-1}$ and boat B moves in a straight line at 32 km $h^{-1}$. The angle between their paths is 70°. Find the distance between the boats after 2.5 hours.

**22.** [Maximum mark: 6] ***[with GDC]***

The following diagram shows the triangle ABC.

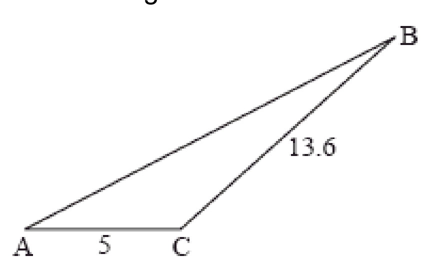



***diagram not to scale***

The angle at C is obtuse, AC = 5 cm, BC = 13.6 cm and the area is 20 $cm^2$.

(a) Find $A\hat{C}B$. [3]

(b) Find AB. [3]

**23.** [Maximum mark: 6] ***[with GDC]***

In a triangle ABC, AB = 4 cm, AC = 3 cm and the area of the triangle is 4.5 cm$^2$.

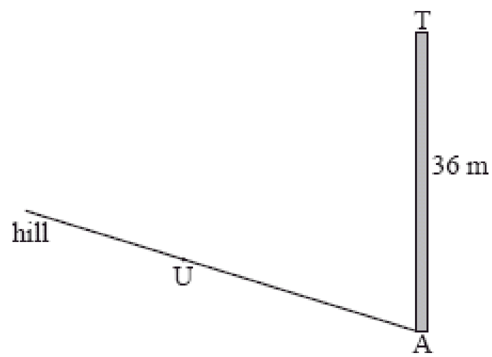
Find the **two** possible values of the angle $\text{B}\hat{\text{A}}\text{C}$.

**24.** [Maximum mark: 6] ***[without GDC]***

In triangle PQR, PQ is 10 cm, QR is 8 cm and angle PQR is acute. The area of the triangle is 20 cm$^2$. Find the size of angle $\text{P}\hat{\text{Q}}\text{R}$.

**25.** [Maximum mark: 7] ***[with GDC]***

There is a vertical tower TA of height 36 m at the base A of a hill. A straight path goes up the hill from A to a point U. This information is represented by the following diagram.




The path makes a 4° angle with the horizontal.

The point U on the path is 25 m away from the base of the tower.

The top of the tower is fixed to U by a wire of length $x$ m.

(a) Complete the diagram, showing clearly all the information above. [3]

(b) Find $x$. [4]

**26.** [Maximum mark: 8] ***[with GDC]***

The diagram below shows a triangle ABD with AB = 13 cm and AD = 6.5 cm.

Let C be a point on the line BD such that BC = AC = 7 cm.

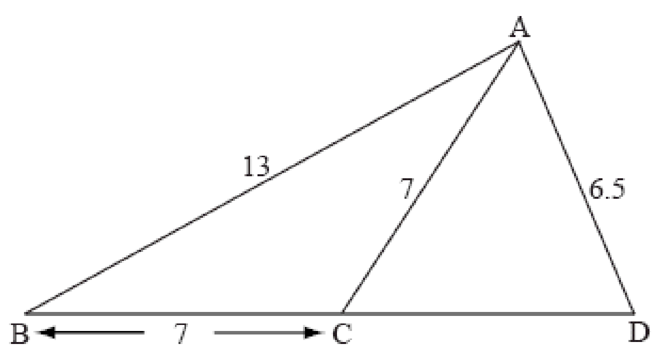


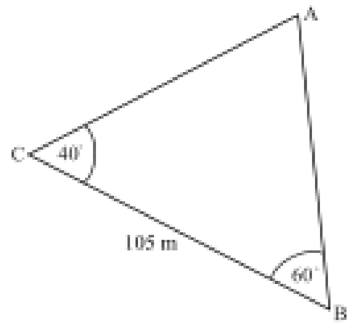

***diagram not to scale***

(a) Find the size of angle ACB. [3]

(b) Find the size of angle CAD. [5]

**27.** [Maximum mark: 6] ***[with GDC]***

The following diagram shows $\Delta ABC$, where $BC = 105$ m, $A\hat{C}B = 40°$, $A\hat{B}C = 60°$




Find the area of the triangle.

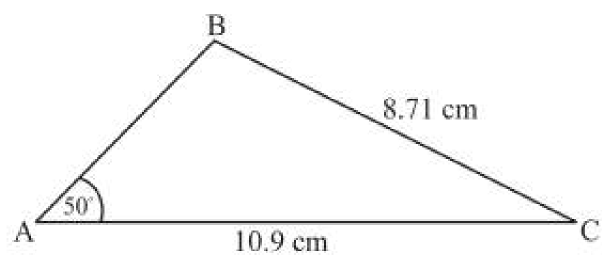
**28.** [Maximum mark: 6] ***[with GDC]***

In the triangle ABC, $\hat{A} = 30°$, BC = 3 and AB = 5. Find the two possible values of $\hat{B}$.

**29.** [Maximum mark: 6] ***[with GDC]***

In the **obtuse-angled** triangle $ABC$, $AC = 10.9\text{ cm}$, $BC = 8.71\text{ cm}$ and $B\hat{A}C = 50°$.

Not to scale




Find the area of triangle $ABC$.

**30.** [Maximum mark: 6] ***[with GDC]***

Triangle $ABC$ has $\hat{C} = 42°$, $BC = 1.74\text{ cm}$, and area $1.19\text{ cm}^2$.

(a) Find $AC$. [3]

(b) Find $AB$. [3]

**31*.** [Maximum mark: 6] ***[with GDC]***

In the triangle $ABC$, $\hat{A} = 30°$, $a = 5$ and $c = 7$. Find the difference in area between the two possible triangles for $ABC$.

**32*.** [Maximum mark: 7] ***[with GDC]***

In a triangle *ABC*, $A\hat{B}C = 30°$, $AB = 6cm$, $AC = 3\sqrt{2}$ *cm*. Find the possible areas of the triangle.

**33*.** [Maximum mark: 6] ***[with GDC]***

In a triangle $\text{ABC}$, $\text{A}\hat{\text{B}}\text{C}=30°$, $\text{AB}=6\,\text{cm}$, $\text{AC}=3\sqrt{2}\,\text{cm}$. Find the possible lengths of $[\text{BC}]$.

***METHOD A: Use Sine rule.***

***METHOD B: Use Cosine rule.***

**34*.** [Maximum mark: 7] ***[with GDC]***

In a triangle $\text{ABC}$, $\hat{A}=35°$, $\text{BC}=4\,\text{cm}$ and $\text{AC}=6.5\,\text{cm}$. Find the possible values of $\hat{\text{B}}$ and the corresponding values of $\text{AB}$.

**35*.** [Maximum mark: 6] ***[with GDC]***

. Triangle $\text{ABC}$ has $\text{AB}=8\,\text{cm}$, $\text{BC}=6\,\text{cm}$, $\text{B}\hat{\text{A}}\text{C}=20°$. Find the smallest possible area of $\Delta\text{ABC}$.

**36*.** [Maximum mark: 6] ***[with GDC]***

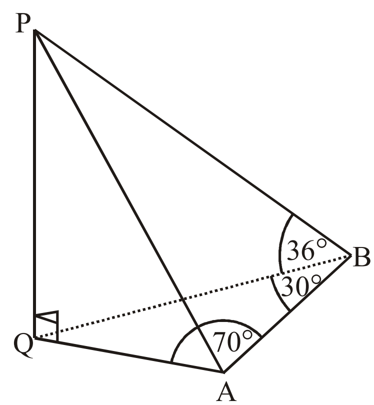
In triangle $\text{ABC}$, $\text{A}\hat{\text{B}}\text{C}=31°$, $\text{AC}=3\,\text{cm}$, $\text{BC}=5\text{cm}$. Calculate the possible lengths of $[\text{AB}]$.

**37*.** [Maximum mark: 7] ***[with GDC]***

Consider triangle $\text{ABC}$ with $\text{B}\hat{\text{A}}\text{C}=37.8°$, $\text{AB}=8.75$ and $\text{BC}=6$. Find $\text{AC}$.

**38*.** [Maximum mark: 4] ***[with GDC]***

The diagram shows a vertical pole PQ, which is supported by two wires fixed to the horizontal ground at A and B.




| |
|---|
| $\text{BQ}=40\text{ m}$ |
| $\text{P}\hat{\text{B}}\text{Q}=36°$ |
| $\text{B}\hat{\text{A}}\text{Q}=70°$ |
| $\text{A}\hat{\text{B}}\text{Q}=30°$ |

Find

(a) the height of the pole, PQ; [2]

(b) the distance between A and B. [2]

**39*.** [Maximum mark: 7] ***[with GDC]***

A ship leaves port A on a bearing of 030°. It sails a distance of 25 km to point B.

At B, the ship changes direction to a bearing of 100°. It sails a distance of 40 km to reach point C. This information is shown in the diagram below.

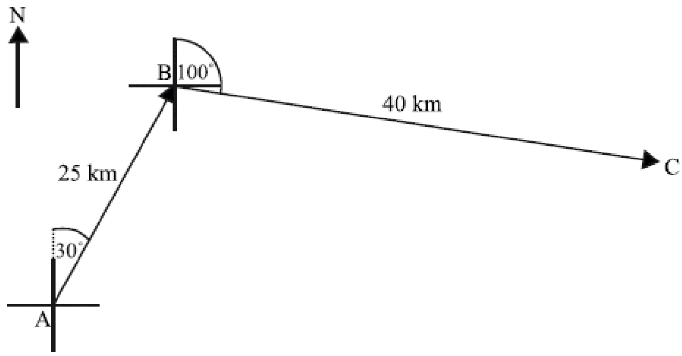



***diagram not to scale***

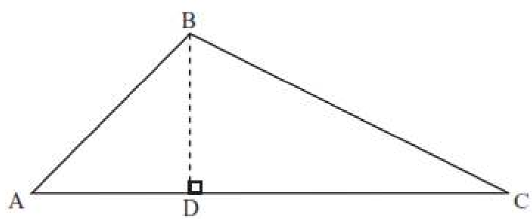
A second ship leaves port A and sails directly to C.

(a) Find the distance the second ship will travel. [4]

(b) Find the bearing of the course taken by the second ship. [3]

**40*.** [Maximum mark: 6] ***[without GDC]***

In triangle $\mathrm{ABC}$, $\mathrm{BC} = a$, $\mathrm{AC} = b$, $\mathrm{AB} = c$ and $[\mathrm{BD}]$ is perpendicular to $[\mathrm{AC}]$.




(a) Show that $\mathrm{BD} = c \sin A$. [1]

(b) Show that $\mathrm{CD} = b - c\cos A$. [2]

(c) **Hence**, by using Pythagoras' Theorem in the triangle $\mathrm{BCD}$, prove the cosine rule for the triangle $\mathrm{ABC}$. [3]

**41**.** [Maximum mark: 7] ***[without GDC]***

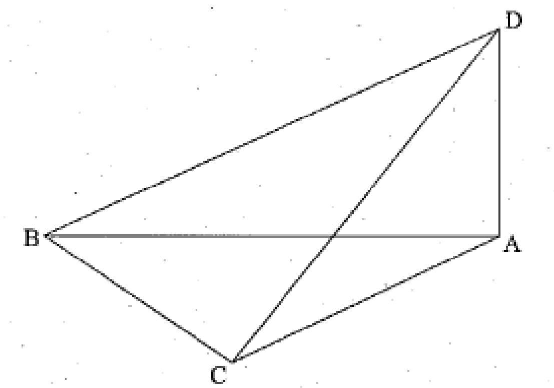
In triangle $\mathrm{ABC}$, $\mathrm{BC} = a$, $\mathrm{AC} = b$, $\mathrm{AB} = c$ and $\mathrm{A\hat{B}C} = 60°$.

Use the cosine rule to show that $c = \frac{1}{2}a \pm \sqrt{b^2 - \frac{3}{4}a^2}$.

**42**.** [Maximum mark: 7] ***[with GDC]***

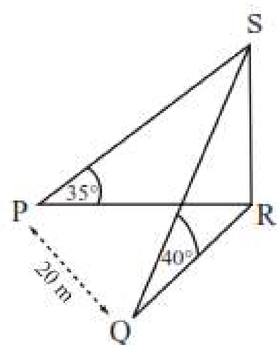
The following three dimensional diagram shows the four points A,B,C and D.

A,B,C are in the same horizontal plane and AD is vertical. The angle ABC is 45$^\circ$ , and BC = 50m. The angle of elevation from point B to point D is 30$^\circ$, while the angle of elevation from point C to point D is 20$^\circ$.




Using the cosine rule in the triangle ABC, or otherwise, find AD.

**43**.** [Maximum mark: 7] ***[with GDC]***



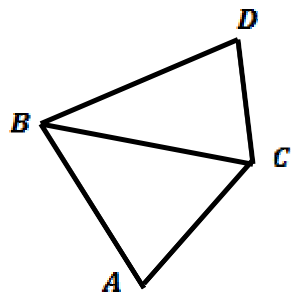

The above 3-dimensional diagram shows the points $\mathrm{P}$ and $\mathrm{Q}$ which are respectively west and south-west of the base $\mathrm{R}$ of a vertical flagpole $\mathrm{RS}$ on horizontal ground. The angles of elevation of the top $\mathrm{S}$ of the flagpole from $\mathrm{P}$ and $\mathrm{Q}$ are respectively $35°$ and $40°$, and $\mathrm{PQ} = 20\,\mathrm{m}$.

Determine the height of the flagpole.

## B. Past paper questions (*LONG*)

**44.** [Maximum mark: 18] ***[with GDC]***

Consider the following diagram




$AB = 7$

$AC = 5$

$\hat{A} = 60^\circ$

$\hat{D} = 80^\circ$

$D\hat{B}C = 30^\circ$

(a) Find the length of the side *BD*. [6]

(b) Find the area of the quadrilateral *ABDC*. [4]

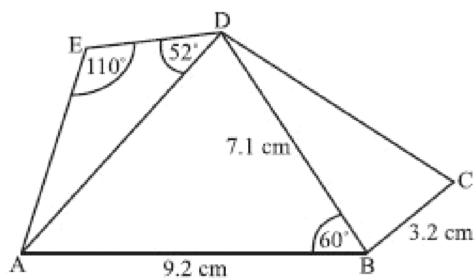
(c) Find the perimeter of the quadrilateral *ABDC*. [3]

It is given that the bearing from *B* to *D* is 70°.

(d) Find (i) the bearing from *B* to *A*. (ii) the bearing from *A* to *B*. [5]

**45.** [Maximum mark: 21] ***[with GDC]***

The following diagram shows a pentagon ABCDE, with AB = 9.2 cm, BC = 3.2 cm, BD = 7.1 cm, $A\hat{E}D = 110^\circ$, $A\hat{D}E = 52^\circ$ and $A\hat{B}D = 60^\circ$.




(a) Find AD. [4]

(b) Find DE. [4]

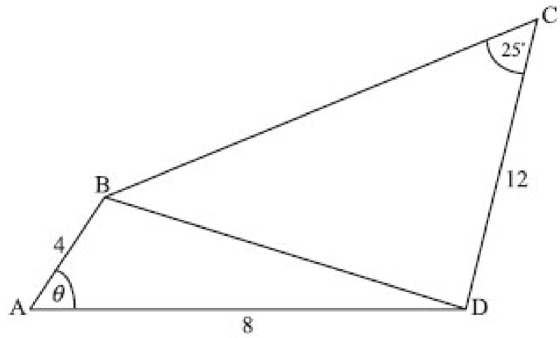
(c) The area of triangle BCD is $5.68\ \text{cm}^2$. Find $D\hat{B}C$. [4]

(d) Find AC. [4]

(e) Find the area of quadrilateral ABCD. [5]

**46.** [Maximum mark: 16] ***[with GDC]***

The diagram below shows a quadrilateral ABCD. AB = 4, AD = 8, CD =12, $B\hat{C}D = 25°$, $B\hat{A}D = \theta$.




(a) Use the cosine rule to show that BD = $4\sqrt{5-4\cos\theta}$. [2]

Let $\theta = 40°$.

(b) (i) Find the value of $\sin C\hat{B}D$.

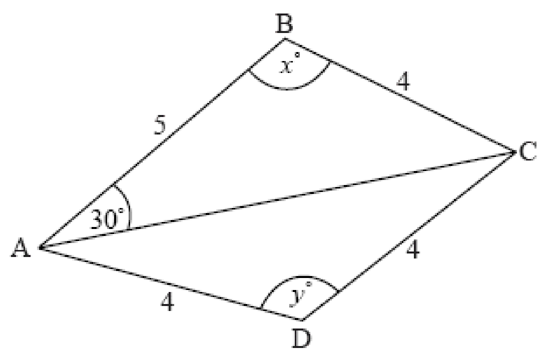
(ii) Find the two possible values for the size of $C\hat{B}D$.

(iii) Given that $C\hat{B}D$ is an acute angle, find the perimeter of ABCD. [12]

(c) Find the area of triangle ABD. [2]

**47.** [Maximum mark: 14] ***[with GDC]***

The diagram below shows a quadrilateral ABCD with obtuse angles $A\hat{B}C$ and $A\hat{D}C$.




AB = 5 cm, BC = 4 cm, CD = 4 cm, AD = 4 cm, $B\hat{A}C = 30°$, $A\hat{B}C = x°$, $A\hat{D}C = y°$.

(a) Use the cosine rule to show that AC = $\sqrt{41-40\cos x}$. [1]

(b) Use the sine rule in triangle ABC to find another expression for AC. [2]

(c) (i) Hence, find $x$, giving your answer to two decimal places.

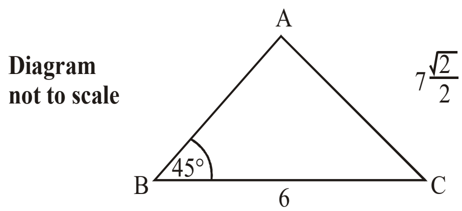
(ii) Find AC. [6]

(d) (i) Find $y$.

(ii) Hence, or otherwise, find the area of triangle ACD. [5]

**48.** [Maximum mark: 10] ***[with GDC]***

The diagram shows a triangle $\mathrm{ABC}$ in which $\mathrm{AC} = 7\dfrac{\sqrt{2}}{2}$, $\mathrm{BC} = 6$, $\mathrm{A\hat{B}C} = 45°$.




(a) Use the fact that $\sin 45° = \dfrac{\sqrt{2}}{2}$ to show that $\sin \mathrm{B\hat{A}C} = \dfrac{6}{7}$. [2]

The point D is on (AB), between A and B, such that $\sin \mathrm{B\hat{D}C} = \dfrac{6}{7}$.

(b) (i) Write down the value of $\mathrm{B\hat{D}C} + \mathrm{B\hat{A}C}$.

(ii) Calculate the angle BCD.

(iii) Find the length of [BD]. [6]

(c) Show that $\dfrac{\text{Area of } \Delta\mathrm{BDC}}{\text{Area of } \Delta\mathrm{BAC}} = \dfrac{\mathrm{BD}}{\mathrm{BA}}$. [2]

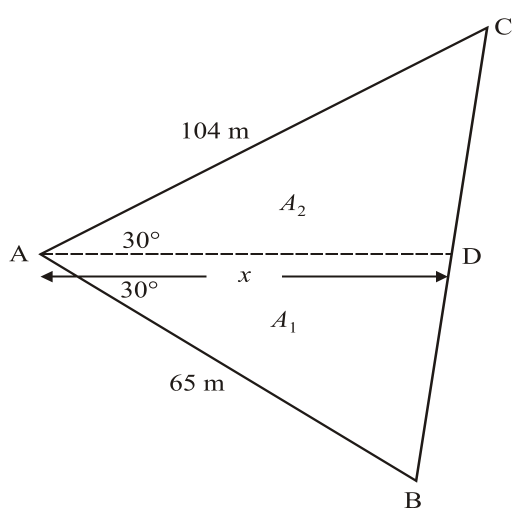
**49.** [Maximum mark: 18] ***[with GDC]***

A farmer owns a triangular field ABC. One side of the triangle, [AC], is 104 m, a second side, [AB], is 65 m and the angle between these two sides is 60°.

(a) Use the cosine rule to calculate the length of the third side of the field. [3]

(b) Given that $\sin 60° = \dfrac{\sqrt{3}}{2}$, find the area of the field in the form $p\sqrt{3}$ where $p \in Z$. [3]

Let D be a point on [BC] such that [AD] bisects the 60° angle. The farmer divides the field into two parts $A_1$ and $A_2$ by constructing a straight fence [AD] of length $x$ metres, as shown on the diagram below.



(c) (i) Show that the area of $A_1$ is given by $\frac{65x}{4}$.

(ii) Find a similar expression for the area of $A_2$.

(iii) **Hence**, find the value of $x$ in the form $q\sqrt{3}$, where $q \in Z$. [7]

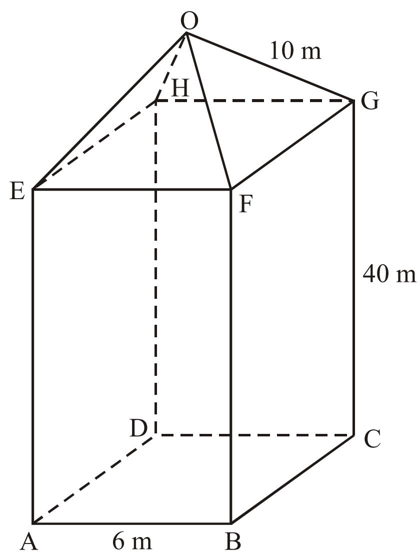
(d) (i) Explain why $\sin A\hat{D}C = \sin A\hat{D}B$.

(ii) Use the result of part (i) and the sine rule to show that $\frac{BD}{DC} = \frac{5}{8}$. [5]

**50.** [Maximum mark: 14] ***[with GDC]***

An office tower is in the shape of a cuboid with a square base. The roof of the tower is in the shape of a square based right pyramid.

The diagram shows the tower and its roof with dimensions indicated. The diagram is **not** drawn to scale.




(a) Calculate, correct to three significant figures,

(i) the size of the angle between OF and FG; [3]

(ii) the shortest distance from O to FG; [2]

(iii) the total surface area of the four triangular sections of the roof; [3]

(iv) the size of the angle between the slant height of the roof and the plane EFGH; [2]

(v) the height of the tower from the base to O. [2]

A parrot's nest is perched at a point, P, on the edge, BF, of the tower. A person at the point A, outside the building, measures the angle of elevation to point P to be 79°.

(b) Find, correct to three significant figures, the height of the nest from the base of the tower. [2]

**51*.** [Maximum mark: 16] ***[with GDC]***

In the diagram below, the points O(0, 0) and A(8, 6) are fixed. The angle $\mathrm{O\hat{P}A}$ varies as the point P( $x$ ,10) moves along the horizontal line $y = 10$.

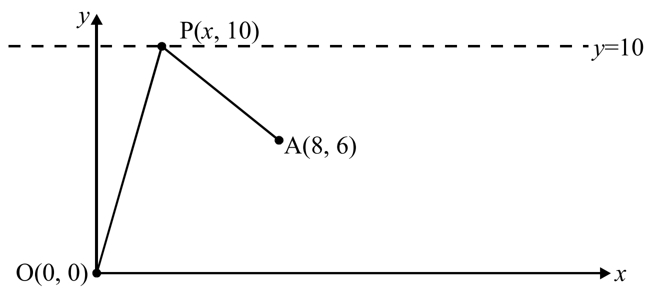



**Diagram to scale**

(a) (i) Show that $\mathrm{AP} = \sqrt{x^2 - 16x + 80}.$

(ii) Write down a similar expression for OP in terms of $x$. [2]

(b) Hence, show that $\cos \mathrm{O\hat{P}A} = \dfrac{x^2 - 8x + 40}{\sqrt{\{(x^2 - 16x + 80)(x^2 + 100)\}}}$, [3]

(c) Find, in degrees, the angle $\mathrm{O\hat{P}A}$ when $x = 8$. [2]

(d) Find the positive value of $x$ such that $\mathrm{O\hat{P}A} = 60°$. [4]

Let the function $f$ be defined by

$$f(x) = \cos \mathrm{O\hat{P}A} = \frac{x^2 - 8x + 40}{\sqrt{\{(x^2 - 16x + 80)(x^2 + 100)\}}}, \quad 0 \le x \le 15.$$ [4]

(e) Consider the equation $f(x) = 1$.

(i) Explain, in terms of the position of the points O, A, and P, why this equation has a solution.

(ii) Find the **exact** solution to the equation. [5]

**52*.** [Maximum mark: 16] ***[with GDC]***

The points P, Q, R are three markers on level ground, joined by straight paths PQ, QR, PR as shown in the diagram. QR = 9 km, $P\hat{Q}R = 35°$, $P\hat{R}Q = 25°$.

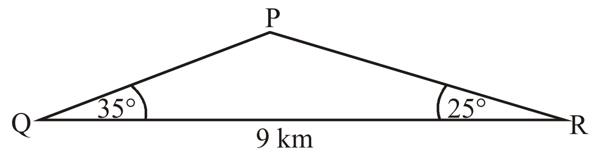
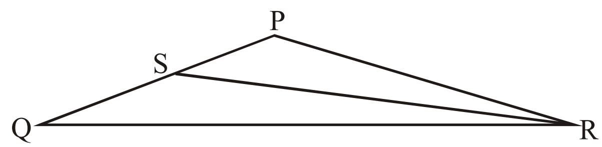



**Diagram not to scale**

(a) Find the length PR. [3]

(b) Tom sets out to walk from Q to P at a steady speed of 8 km $h^{-1}$. At the same time, Alan sets out to jog from R to P at a steady speed of $a$ km $h^{-1}$. They reach P at the same time. Calculate the value of $a$. [7]

(c) The point S is on [PQ], such that RS = 2QS, as shown in the diagram.




Find the length QS. [6]