INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

Math AA

EXERCISES [Math-AA 3.17]

PLANES

Compiled by Christos Nikolaidis

O. Practice questions

1. [Maximum mark: 6] ***[without GDC]***

The **vector equation** of the plane Π_1 is $\boldsymbol{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\5\\6\end{pmatrix}+\mu\begin{pmatrix}7\\8\\9\end{pmatrix}$.

(a) (i) Write down the coordinates of a particular point in the plane Π_1.

(ii) Write down two vectors which are **parallel** to the plane Π_1. [3]

The **Cartesian equation** of the plane Π_2 is $2x+3y+4z=9$.

(b) (i) Write down a **normal** vector to the plane Π_2.

(ii) Determine whether the points A(1,1,1) and B(3,0,1) lie in the plane Π. [3]

2. [Maximum mark: 7] ***[without GDC]***

The equation of a plane can be given in three different forms:

- The **vector equation** of a plane has the form $\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{b}+\mu\boldsymbol{c}$
- The **Cartesian equation** of a plane has the form $ax+by+cz=d$.
- The equation of a plane **using the normal vector** is $\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}$.

Consider the plane that passes through the point A(1,1,1) and is parallel to the vectors

$\boldsymbol{b}=\begin{pmatrix}2\\-3\\4\end{pmatrix}$ and $\boldsymbol{c}=\begin{pmatrix}3\\-4\\5\end{pmatrix}$. It is also given that $\boldsymbol{b}\times\boldsymbol{c}=\begin{pmatrix}1\\2\\1\end{pmatrix}$.

(a) Write down the v**ector equation** of Π. [2]

(b) Write down the Cartesian **equation** of Π. [2]

(c) For the equation $\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}$, state what the vectors $\boldsymbol{r}$, $\boldsymbol{n}$ and $\boldsymbol{a}$ represent.

Confirm that $\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}$ is in fact the Cartesian equation found in (b). [3]

3. [Maximum mark: 8] ***[without GDC]***

Let the **vector equation** of plane Π be $\boldsymbol{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\1\end{pmatrix}+\mu\begin{pmatrix}4\\5\\6\end{pmatrix}$

(a) Find the coordinates of the following points on Π :

Point	Corresponding to the parameters	Coordinates
A	$\lambda=0$, $\mu=0$	$(1,2,3)$
B	$\lambda=1$, $\mu=0$	
C	$\lambda=0$, $\mu=1$	
D	$\lambda=1$, $\mu=1$	
E	$\lambda=3$, $\mu=2$	

[4]

(c) Find a vector which is perpendicular to the plane Π (i.e. **normal**). [2]

(d) Hence, find the **Cartesian equation** of Π (i.e. in the form $ax+by+cz=d$). [2]

4. [Maximum mark: 8] ***[without GDC]***

The **Cartesian equation** of plane Π is $x+2y+3z=6$

(a) Find the coordinates of the following points in plane Π .

Point	Corresponding to the variables	Coordinates
A	$x=0$, $y=0$	$(0,0,2)$
B	$x=0$, $z=0$	
C	$y=0$, $z=0$	
D	$x=1$, $y=1$	

[3]

(b) Write down the **normal** vector to the plane Π . [1]

(c) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [2]

(d) Hence, write down a **vector equation** of Π (i.e. in the form $\boldsymbol{r}=a+\lambda\boldsymbol{b}+\mu\boldsymbol{c}$). [2]

5. [Maximum mark: 4] ***[without GDC]***

The equation $\boldsymbol{r}\cdot\begin{pmatrix}2\\3\\6\end{pmatrix}=18$ represents a plane Π .

(a) Write down **the Cartesian equation** of Π (i.e. in the form $ax+by+cz=d$). [1]

(b) Plane Π intersects the axes Ox, Oy and Oz at points A, B and C respectively.

Find the coordinates of A , B and C . [3]

6. [Maximum mark: 6] ***[without GDC]***

The **vector equations** of two planes are given below

$$\Pi_1: \quad \boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \qquad \Pi_2: \quad \boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Express Π_1 and Π_2 in the Cartesian form $ax + by + cz = d$.

7. [Maximum mark: 6] ***[without GDC]***

Consider the standard vectors $\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(a) Find the following cross products.

$\boldsymbol{i} \times \boldsymbol{j}$	
$\boldsymbol{j} \times \boldsymbol{k}$	
$\boldsymbol{k} \times \boldsymbol{i}$	

[3]

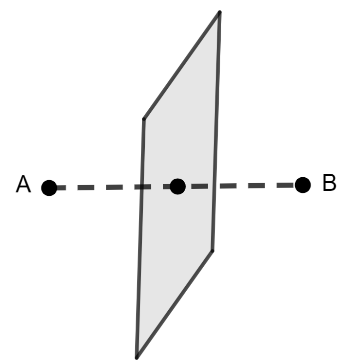
(b) Complete the following table with the equations of the three standard planes Oxy , Oyz , and Ozx .

Plane	**Vector equation** (in the form $\boldsymbol{r} = a + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$)	**Cartesian equation** (in the form $ax + by + cz = d$)
Oxy		
Oyz		
Ozx		

[3]

8. [Maximum mark: 8] ***[without GDC]***

The diagram shows the points A(1,3,5) and B(3,5,1) and the **perpendicular bisector plane** Π of the line segment [AB] (that is the plane which is perpendicular to the line (AB) and passes through the midpoint of [AB])



Find in the Cartesian form $ax+by+cz=d$, where a,b,c,d are integers,

(a) the equation of plane Π. [5]

(b) the equations of the planes Π_1 and Π_2 which are parallel to Π and pass through the points A and B respectively. [3]

9. [Maximum mark: 12] ***[without GDC]***

The triangle ABC with $\mathrm{A}(2, 1, -5)$, $\mathrm{B}(1, 1, 1)$ and $\mathrm{C}(3, -1, -2)$ lies in a plane Π.

(a) Find the vectors $\overrightarrow{\mathrm{AB}}$, $\overrightarrow{\mathrm{AC}}$ and $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}}$. [4]

Hence,

(b) find the **vector equation** of plane Π. [2]

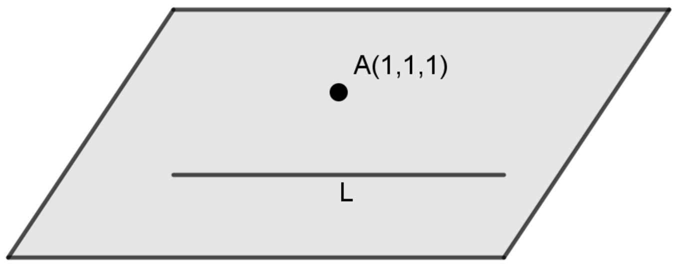
(c) find the **Cartesian equation** of plane Π. [3]

(d) find the area of the parallelogram determined by the vectors $\overrightarrow{\mathrm{AB}}, \overrightarrow{\mathrm{AC}}$. [2]

(e) find the area of the triangle ABC. [1]

10*. [Maximum mark: 9] ***[without GDC]***

Plane Π contains the point $\mathrm{A}(1, 1, 1)$ and the line L: $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$.



(a) Find the Cartesian equation of plane Π. [5]

(b) Find the Cartesian equation of plane Π_1 which is perpendicular to Π and contains L. [4]

11. [Maximum mark: 10] **[without GDC]**

Consider the plane Π: $x + 2y + 3z = 6$

(a) Explain why the plane $2 + 4y + 6z = 11$ is parallel to the plane Π. [1]

(b) Show that the line $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is parallel to the plane Π (not lining in Π). [3]

(c) Show that the line $\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ lies in the plane Π. [3]

A. Exam style questions (SHORT)

12. [Maximum mark: 10] ***[without GDC]***

Consider the point $\mathrm{A}(2, 1, 5)$ and the vectors $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$.

(a) Write down the vector equation of the plane Π that passes through the point A and is parallel to the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$. [2]

(b) Find the vector $\boldsymbol{b} \times \boldsymbol{c}$. [3]

(c) The equation of the plane Π can be written in the form $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$. Write down the vectors $\boldsymbol{n}$ and $\boldsymbol{a}$. [2]

(d) **Hence**, find the Cartesian equation $ax + by + cz = d$ of the plane Π. [3]

13. [Maximum mark: 6] ***[without GDC]***

Consider the point $\mathrm{A}(1, -1, 4)$ and the line L_2 with equation $\boldsymbol{r} = 2\boldsymbol{i}+4\boldsymbol{j}+7\boldsymbol{k} + t(2\boldsymbol{i}+\boldsymbol{j}+3\boldsymbol{k})$ where $t \in \mathbb{R}$. Find the Cartesian equation of the plane that contains both the line L_2 and point A.

14. [Maximum mark: 6] ***[without GDC]***

Find an equation of the plane containing the two lines

$$x - 1 = \frac{1-y}{2} = z - 2 \text{ and } \frac{x+1}{3} = \frac{2-y}{3} = \frac{z+2}{5}.$$

B. Exam style questions (LONG)

15. [Maximum mark: 21] ***[with GDC]***

Consider the points $\mathrm{A}(2, -1, 0)$, $\mathrm{B}(3, 0, 1)$ and $\mathrm{C}(1, m, 2)$, where $m \in \mathbb{Z}, m < 0$.

(a) (i) Find the scalar product $\overrightarrow{\mathrm{BA}} \cdot \overrightarrow{\mathrm{BC}}$.

(ii) Hence, given that $\mathrm{A\hat{B}C} = \arccos \frac{\sqrt{2}}{3}$, show that $m = -1$. [6]

(b) Determine the Cartesian equation of the plane ABC. [4]

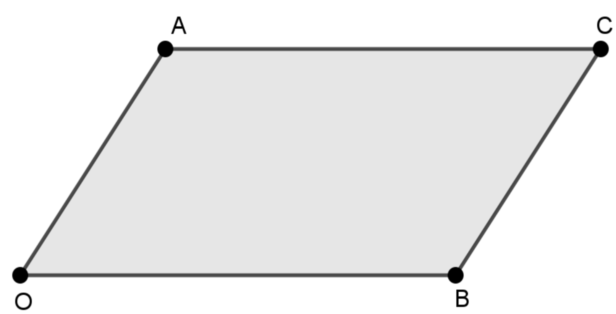
(c) Find the area of the triangle ABC. [3]

(d) (i) The line L is perpendicular to plane ABC and passes through A. Find a vector equation of L.

(ii) The point $\mathrm{D}(6, -7, 2)$ lies on L. Find the volume of the pyramid ABCD. [8]

16. [Maximum mark: 12] ***[without GDC]***

Consider the parallelogram OACB with $\mathrm{A}(2, 1, -2)$, $\mathrm{B}(2, -1, -1)$, where O is the origin.



(a) Find the coordinates of C. [2]

(b) Find an equation for the plane OACB. [4]

(c) Find the area of the parallelogram OACB. [2]

(d) Find the Cartesian equation for the plane which is perpendicular to the rectangle OACB and contains the line (OB). [4]