INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

Math AA

EXERCISES [Math-AA 3.16]

CROSS PRODUCT

Compiled by Christos Nikolaidis

O. Practice questions

1. [Maximum mark: 8] ***[without GDC]***

Let $\boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(a) Find $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{b} \cdot \boldsymbol{a}$. [2]

(b) Find $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{b} \times \boldsymbol{a}$. [3]

(c) Show that $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$. [3]

2. [Maximum mark: 10] ***[without GDC]***

Let $\boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$.

(a) Find $(\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$. [2]

(b) Find $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$ and $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$. [6]

(c) Find $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$. [2]

3. [Maximum mark: 8] ***[without GDC]***

Consider the points $A(1,2,3)$, $B(3,0,5)$, $C(4,3,5)$).

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$ [4]

(b) Find the area of the triangle ABC. [2]

The point $D(5,7,3)$ does not lie in the same plane with the triangle ABC.

The volume of the tetrahedron $ABCD$ is given by $\mathrm{V} = \frac{1}{6}\left|\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})\right|$.

(c) Find the volume V. [2]

A. Exam style questions (SHORT)

4. [Maximum mark: 5] ***[with / without GDC]***

The position vectors of points P and Q are: $\boldsymbol{p} = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{q} = \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$

(a) Find the vector product $\boldsymbol{p} \times \boldsymbol{q}$. [3]

(b) Using your answer to part (a), or otherwise, find the area of the parallelogram with two sides $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. [2]

5. [Maximum mark: 5] ***[without GDC]***

(a) Find a vector perpendicular to the two vectors:

$$\overrightarrow{OP} = \vec{i} - 3\vec{j} + 2\vec{k}, \quad \overrightarrow{OQ} = -2\vec{i} + \vec{j} - \vec{k}.$$ [3]

(b) If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are position vectors for the points P and Q, use your answer to part (a), or otherwise, to find the area of the triangle OPQ. [2]

6. [Maximum mark: 5] ***[without GDC]***

Let $\boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -1 \\ p \\ 6 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$.

(a) Find $\boldsymbol{a} \times \boldsymbol{b}$. [3]

(b) Find the value of p, given that $\boldsymbol{a} \times \boldsymbol{b}$ is parallel to $\boldsymbol{c}$. [2]

7. [Maximum mark: 5] ***[without GDC]***

Given that $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = -3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$, find $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$.

8. [Maximum mark: 5] ***[without GDC]***

For the vectors $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$, show that:

(a) $\boldsymbol{a} \times \boldsymbol{b} = -3\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k}$

(b) $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = -(\boldsymbol{b} \bullet \boldsymbol{c})\boldsymbol{a}$

9. [Maximum mark: 4] ***[without GDC]***

Find a vector that is perpendicular to the plane containing the lines L_1, and L_2, with vector equations

$$L_1 : \boldsymbol{r} = \boldsymbol{i} + \boldsymbol{k} + \lambda(2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) \qquad L_2 : \boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k} + \mu(\boldsymbol{j} + 3\boldsymbol{k})$$

10. [Maximum mark: 6] ***[without GDC]***

The parallelogram ABCD has vertices A(3,2,0), B(7,-1,-1), C(10,-3,0) and D(6,0,1).

Calculate the area of the parallelogram.

11. [Maximum mark: 6] ***[with GDC]***

Consider the points A(1, 2, –4), B(1, 5, 0) and C(6, 5, –12).

Find the area of the triangle ABC.

12. [Maximum mark: 6] ***[without GDC]***

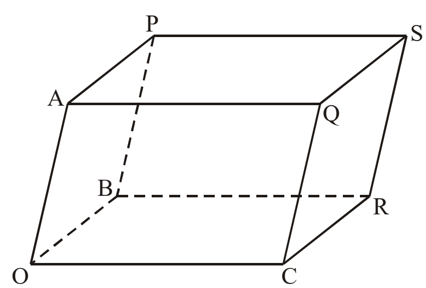
Given that $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{c} = -\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ are the position vectors of the points A, B and C respectively, calculate the area of triangle ABC.

13. [Maximum mark: 6] ***[without GDC]***

Given any two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, show that $|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \bullet \boldsymbol{b})^2$.

14. [Maximum mark: 7] ***[without GDC]***

Three points A, B and C have coordinates (2, 1, –2), (2, –1, –1) and (1, 2, 2) respectively. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, where O is the origin, form three concurrent edges of a parallelepiped OAPBCQSR as shown in the following diagram.



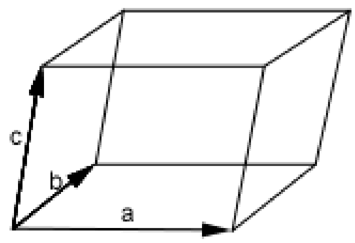
(a) Find the coordinates of P, Q, R and S. [4]

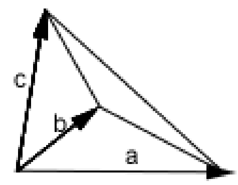
(b) Calculate the volume, V, of the parallelepiped given that $V = |\overrightarrow{OA} \times \overrightarrow{OB} \bullet \overrightarrow{OC}|$ [3]

B. Exam style questions (LONG)

15. [Maximum mark: 13] ***[with / without GDC]***

Three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ (which are not coplanar) define a parallelepiped (or cuboid) and a tetrahedron (or triangular pyramid) as shown in the diagrams below.





It is given that

Volume of parallelepiped $= |(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}|$

Volume of tetrahedron $= \dfrac{1}{6}|(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}|$

Let $\boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$.

(a) Find the area of the parallelogram and the area of triangle defined by $\boldsymbol{a}$ and $\boldsymbol{b}$. [6]

(b) Find the so-called **triple product** $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$ [2]

(c) **Hence** find

(i) the volume of the parallelepiped defined by $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

(ii) the volume of the tetrahedron defined by $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

(iii) the height of the tetrahedron (or of the parallelepiped) corresponding to the base defined by $\boldsymbol{a}$ and $\boldsymbol{b}$ [5]

16. [Maximum mark: 10] ***[without GDC]***

The coordinates of the points P, Q and R are $(4,1,-1)$, $(3,3,5)$ and $(1,0,2c)$ respectively.

The vectors $\overrightarrow{\mathrm{QR}}$ and $\overrightarrow{PR}$ are orthogonal (perpendicular).

(a) Find the value of c. [4]

(b) Evaluate $\overrightarrow{PS} \times \overrightarrow{PR}$. [4]

(c) Find the vector equation of the line *l* which passes through the point Q and is parallel to the vector PR. [2]

17. [Maximum mark: 10] ***[without GDC]***

Consider the points $A(1, 2, 1)$, $B(0, -1, 2)$, $C(1, 0, 2)$ and $D(2, -1, -6)$.

(a) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. [2]

(b) Calculate $\overrightarrow{AB} \times \overrightarrow{BC}$. [3]

(c) Hence, or otherwise find the area of triangle ABC. [2]

(d) Find a set of parametric equations for the line *L* through the point D and perpendicular to the triangle ABC. [3]

18. [Maximum mark: 11] ***[without GDC]***

The points A, B, C, D have the following coordinates

$A : (1, 3, 1)$ $B : (1, 2, 4)$ $C : (2, 3, 6)$ $D : (5, -2, 1)$.

(a) (i) Evaluate the vector product $\overrightarrow{AB} \times \overrightarrow{AC}$, giving your answer in terms of the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

(ii) Find the area of the triangle ABC. [6]

(b) Find the vector equations of the lines

(i) L_1 which passes through D and is parallel to $\overrightarrow{AB}$.

(ii) L_2 which passes through D and A.

(iii) L_3 which passes through D and is perpendicular to the triangle ABC. [5]