INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

Math AA

EXERCISES [Math-AA 3.15]

VECTORS IN KINEMATICS

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O. Practice questions

1. [Maximum mark: 9] ***[without GDC]***

The motion of a particle is given by the equation $\vec{r} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, where t is in seconds.

(a) Find the initial position of the particle and its position after 2 seconds. [2]

(b) Find the velocity and the speed of the particle. [3]

(c) How far from the origin is the particle after 2 seconds? [2]

(d) How far from its initial position is the particle after 2 seconds? [2]

2. [Maximum mark: 6] ***[without GDC]***

A particle is initially at point A(2,3) and it is moving on a straight line, towards B(8,6) with a constant velocity. Complete the following table

Cases	**Vector** $\overrightarrow{AB}$	**Velocity vector**	**Equation of motion**
at point B(8,6) after 1 sec			
at point B(8,6) after 3 sec			

3. [Maximum mark: 7] ***[without GDC]***

Two bodies are moving according to the equations

$$\boldsymbol{r_1} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \text{ and } \boldsymbol{r_2} = \begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

(a) Write down the velocity vectors and find the speeds of the two bodies. [4]

(b) Show that P(5,6,7) is a common point of their paths. [2]

(c) Do the two bodies collide or not? [1]

4. [Maximum mark: 6] ***[without GDC]***

A particle is initially at point A(1,2,3). It is moving on a straight line, with a constant velocity, and reaches point B(5,4,9) after 2 minutes.

(a) Find the vector equation of its motion. [5]

(b) Find its speed. [1]

5. [Maximum mark: 6] ***[without GDC]***

A particle is initially at point A(1,2,3). It is moving with a constant velocity in the direction of the vector $\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ with speed 15 ms^{-1}.

(a) Find its velocity vector. [4]

(b) Write down the vector equation of its motion. [2]

6*. [Maximum mark: 7] ***[without GDC]***

A body starts moving from the point with coordinates (0,10) along a straight line. The vector equation of its motion is

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ 10 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad (t \text{ is measured in seconds and distance in meters})$$

(a) Find the speed of the body. [2]

(b) Determine the time interval and the total time during which the body is within a circle centered at the origin with radius 5. [5]

A. Exam style questions (SHORT)

7. [Maximum mark: 10] ***[without GDC]***

The equations of the motion of two particles are given by

$$\vec{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \text{ and } \vec{r} = \begin{pmatrix} 2 \\ 9 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

(a) Find the point of intersection of their paths. [4]

(b) Explain why the two particles do not collide. [1]

(c) Find the speeds of the two particles. [2]

(d) Find the distance between the particles after 1 sec. [3]

8. [Maximum mark: 6] ***[without GDC]***

A boat moves with constant velocity along a straight line. Its velocity vector is given by

$\boldsymbol{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. At time $t = 0$ it is at the point (−2, 1).

(a) Find the speed of the boat. [2]

(b) Write down a vector equation representing the position of the boat, giving your answer in the form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$. [2]

(c) Find the coordinates of the boat when $t = 2$. [2]

9. [Maximum mark: 6] ***[without GDC]***

A particle is moving with a constant velocity along a line. Its initial position is A(6, –2, 10). After **one second** the particle has moved to B(9, –6, 15).

(a) (i) Find the velocity vector, $\overrightarrow{\mathrm{AB}}$. (ii) Find the speed of the particle. [4]

(b) Write down the equation of the motion of the particle. [2]

10. [Maximum mark: 8] ***[without GDC]***

Car 1 moves in a straight line, starting at point A (0, 12). Its position p seconds after it starts is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 12 \end{pmatrix} + p\begin{pmatrix} 5 \\ -3 \end{pmatrix}$.

(a) Find the position vector of the car after 2 seconds. [1]

Car 2 moves in a straight line starting at point B (14, 0). Its position q seconds after it starts is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 0 \end{pmatrix} + q\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Cars 1 and 2 collide at point P.

(b) (i) Find the value of p and the value of q when the collision occurs.

(ii) Find the coordinates of P. [6]

(c) Do they start at the same time? Explain. [1]

11*. [Maximum mark: 6] ***[with GDC]***

A circular park is centered at the origin (0,0) with a radius of 1 km. A runner starts running outside the park, 2 km north of the park's center, heading along a straight line and passes through the park. The vector equation of his motion is

$$\boldsymbol{r}_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 4 \\ -8 \end{pmatrix} \quad \text{(where } t \text{ is measured in hours and distance in km)}$$

(a) Find the speed of the runner (in 3 s.f.) [2]

(b) Determine the total time in **minutes**, during which the runner is within the park. [4]

B. Exam style questions (LONG)

12. [Maximum mark: 20] ***[with GDC]***

In this question the vectors $\begin{pmatrix}1\\0\end{pmatrix}$ km and $\begin{pmatrix}0\\1\end{pmatrix}$ km represent displacements due east and due north respectively. The point (0, 0) is the position of *Shipple Airport.*

The position vector r_1 of an aircraft *Air One* is given by

$$r_1 = \begin{pmatrix}16\\12\end{pmatrix} + t\begin{pmatrix}12\\-5\end{pmatrix},$$ where t is the time in minutes **since** 12:00.

(a) Show that the *Air One* aircraft

(i) is 20 km from *Shipple Airport* at 12:00; (ii) has a speed of 13 km/min. [4]

(b) Show that a cartesian equation of the path of *Air One* is: $5x + 12y = 224$. [3]

The position vector r_2 of an aircraft *Air Two* is given by

$$r_2 = \begin{pmatrix}23\\-5\end{pmatrix} + t\begin{pmatrix}2.5\\6\end{pmatrix},$$ where t is the time in minutes **since** 12:00.

(c) Find the angle between the paths of the two aircraft. [4]

(d) (i) Find a cartesian equation for the path of *Air Two.*

(ii) Hence find the coordinates of the point where the two paths cross. [5]

(e) The two aircraft are flying at the same height. Show that they do not collide. [4]

13. [Maximum mark: 14] ***[with GDC]***

In this question, a unit vector represents a displacement of 1 metre.

A miniature car moves in a straight line, starting at the point (2, 0).

After t seconds, its position, (x, y), is given by the vector equation $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} + t\begin{pmatrix}0.7\\1\end{pmatrix}$

(a) How far from the point (0, 0) is the car after 2 seconds? [2]

(b) Find the speed of the car. [2]

(c) Obtain the equation of the car's path in the form $ax + by = c$. [2]

Another miniature vehicle, a motorcycle, starts at the point (0, 2), and travels in a straight line with constant speed. The equation of its path is $y = 0.6x + 2,\ x \geq 0$.

Eventually, the two miniature vehicles collide.

(d) Find the coordinates of the collision point. [3]

(e) If the motorcycle left point (0, 2) at the same moment the car left point (2, 0), find the speed of the motorcycle. [5]

14. [Maximum mark: 17] ***[with GDC]***

In this question, distance is in metres, time is in minutes. Two model airplanes are each flying in a straight line. At 13:00 the first model airplane is at the point (3, 2, 7).

Its position vector after t minutes is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \\ 10 \end{pmatrix}.$$

(a) Find the speed of the model airplane. [2]

At 13:00 the second model airplane is at the point (– 5, 10, 23). After two minutes, it is at the point (3, 16, 39).

(b) Show that its position vector after t minutes is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 10 \\ 23 \end{pmatrix} + t \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix}.$$ [3]

(c) The airplanes meet at point Q.

(i) At what time do the airplanes meet?

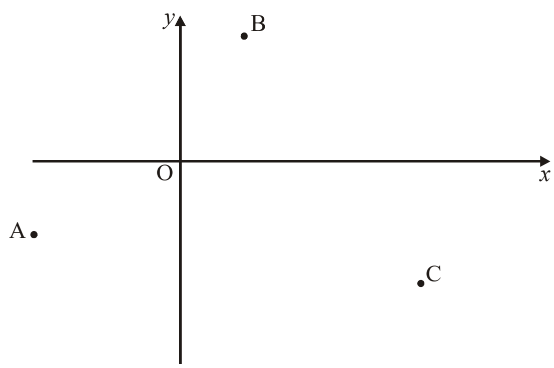
(ii) Find the position of Q. [6]

(d) Find the angle θ between the paths of the two airplanes. [6]

15. [Maximum mark: 17] ***[with GDC]***

In this question the vector $\begin{pmatrix}1\\0\end{pmatrix}$ represents a displacement of 1 km east, the vector $\begin{pmatrix}0\\1\end{pmatrix}$ represents a displacement of 1 km north.

The diagram below shows the positions of towns A, B and C in relation to an airport O, which is at the point (0, 0). An aircraft flies over the three towns at a constant speed of 250 km h^{-1}.



Town A is 600 km west and 200 km south of the airport.

Town B is 200 km east and 400 km north of the airport.

Town C is 1200 km east and 350 km south of the airport.

(a) (i) Find $\overrightarrow{AB}$.

(ii) Show that the vector of length one unit in the direction of $\overrightarrow{AB}$ is $\begin{pmatrix}0.8\\0.6\end{pmatrix}$. [4]

An aircraft flies over town A at 12:00, heading towards town B at 250 km h^{-1}.

Let $\begin{pmatrix}p\\q\end{pmatrix}$ be the velocity vector of the aircraft. Let *t* be the number of hours in flight after 12:00.

The position of the aircraft can be given by the vector equation $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-600\\-200\end{pmatrix}+t\begin{pmatrix}p\\q\end{pmatrix}$.

(b) (i) Show that the velocity vector is $\begin{pmatrix}200\\150\end{pmatrix}$.

(ii) Find the position of the aircraft at 13:00.

(iii) At what time is the aircraft flying over town B? [6]

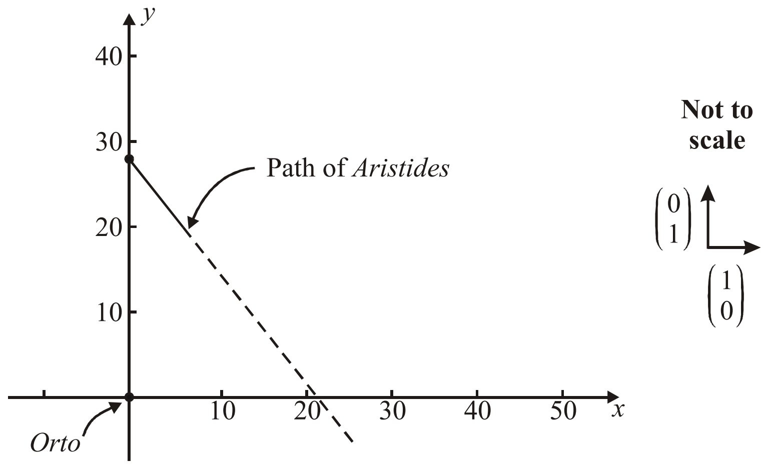
Over town B the aircraft changes direction so it now flies towards town C. It takes five hours to travel the 1250 km between B and C. Over town A the pilot noted that she had $17\ 000$ litres of fuel left. The aircraft uses 1800 litres of fuel per hour when travelling at 250 km h^{-1}. When the fuel gets below 1000 litres a warning light comes on.

(c) How far from town C will the aircraft be when the warning light comes on? [7]

16. [Maximum mark: 20] ***[with GDC]***

In this question the vector $\begin{pmatrix}1\\0\end{pmatrix}$ km represents a displacement due east, the vector $\begin{pmatrix}0\\1\end{pmatrix}$ km represents a displacement due north.

The diagram shows the path of the oil-tanker *Aristides* relative to the port of *Orto*, which is situated at the point (0, 0).



The position of the *Aristides* is given by the vector equation $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\28\end{pmatrix}+t\begin{pmatrix}6\\-8\end{pmatrix}$ at a time t hours after 12:00.

(a) Find the position of the *Aristides* at 13:00. [2]

(b) Find (i) the velocity vector; (ii) the speed of the *Aristides*. [4]

(c) Find a cartesian equation for the path of the *Aristides* in the form $ax + by = c$. [4]

Another ship, the cargo-vessel *Boadicea,* is stationary, with position vector $\begin{pmatrix}18\\4\end{pmatrix}$ km.

(d) Show that the two ships will collide and find the time of collision. [4]

To avoid collision, the *Boadicea* starts to move at 13:00 with velocity vector $\begin{pmatrix}5\\12\end{pmatrix}\text{kmh}^{-1}$

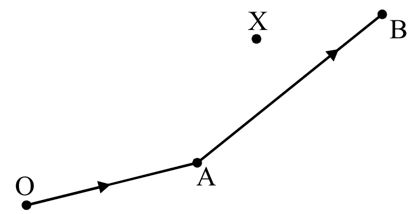
(e) Show that the position of the *Boadicea* for $t \geq 1$ is given by $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}13\\-8\end{pmatrix}+t\begin{pmatrix}5\\12\end{pmatrix}$ [2]

(f) Find how far apart the two ships are at 15:00. [4]

17. [Maximum mark: 18] ***[with GDC]***

The diagram below shows the positions of towns O, A, B and X.

Diagram not to scale



Town A is 240 km East and 70 km North of O.

Town B is 480 km East and 250 km North of O.

Town X is 339 km East and 238 km North of O.

An airplane flies at a constant speed of $300\ \text{km h}^{-1}$ from O towards A.

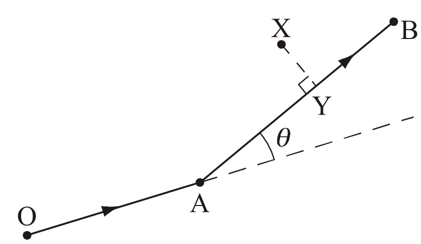
(a) (i) Show that a unit vector in the direction of $\overrightarrow{\text{OA}}$ is $\begin{pmatrix} 0.96 \\ 0.28 \end{pmatrix}$.

(ii) Write down the velocity vector for the airplane in the form $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

(iii) How long does it take for the airplane to reach A? [5]

At A the airplane changes direction and flies towards B. The angle between the original direction and the new direction is θ as shown in the following diagram. This diagram also shows the point Y, between A and B, where the airplane comes closest to X.

Diagram not to scale



(b) Use the scalar product of two vectors to find the value of θ in degrees. [4]

(c) (i) Write down the vector $\overrightarrow{\text{AX}}$.

(ii) Show that the vector $\boldsymbol{n} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ is perpendicular to $\overrightarrow{\text{AB}}$.

(iii) By finding the projection of $\overrightarrow{\text{AX}}$ in the direction of $\boldsymbol{n}$, find the distance XY. [6]

(d) How far is the airplane from A when it reaches Y? [3]

18. [Maximum mark: 20] ***[with GDC]***

In this question the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ km represents a displacement due east, the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ km represents a displacement due north.

Two crews of workers are laying an underground cable in a north–south direction across a desert. At 06:00 each crew sets out from their base camp which is situated at the origin (0, 0). One crew is in a Toyundai vehicle and the other in a Chryssault vehicle.

Toyundai has velocity vector $\begin{pmatrix} 18 \\ 24 \end{pmatrix}$ km h^{-1}, Chryssault has velocity vector $\begin{pmatrix} 36 \\ -16 \end{pmatrix}$ km h^{-1}.

(a) Find the speed of each vehicle. [2]

(b) (i) Find the position vectors of each vehicle at 06:30.

(ii) Hence, or otherwise, find the distance between the vehicles at 06:30. [5]

(c) At this time (06:30) the Chryssault stops and its crew begin their day's work, laying cable in a northerly direction. The Toyundai continues travelling in the same direction at the same speed until it is exactly north of the Chryssault. The Toyundai crew then begin their day's work, laying cable in a southerly direction. At what time does the Toyundai crew begin laying cable? [4]

(d) Each crew lays an average of 800 m of cable in an hour. If they work non-stop until their lunch break at 11:30, what is the distance between them at this time? [4]

(e) How long would the Toyundai take to return to base camp from its lunch-time position, assuming it travelled in a straight line and with the same average speed as on the morning journey? (Give your answer to the nearest minute.) [5]

19. [Maximum mark: 15] ***[with GDC]***

In this question, distance is in kilometers, time is in hours. A balloon is moving at a constant height with a speed of 18 km h^{-1}, in the direction of $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$

At time $t = 0$, the balloon is at point B with coordinates $(0, 0, 5)$.

(a) Show that the position vector $\boldsymbol{b}$ of the balloon at time t is given by

$$\boldsymbol{b} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 10.8 \\ 14.4 \\ 0 \end{pmatrix}.$$ [6]

At time $t = 0$, a helicopter goes to deliver a message to the balloon. The position vector $\boldsymbol{h}$ of the helicopter at time t is given by

$$\boldsymbol{h} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 49 \\ 32 \\ 0 \end{pmatrix} + t\begin{pmatrix} -48 \\ -24 \\ 6 \end{pmatrix}$$

(b) (i) Write down the coordinates of the starting position of the helicopter.

(ii) Find the speed of the helicopter. [4]

(c) The helicopter reaches the balloon at point R.

(i) Find the time the helicopter takes to reach the balloon.

(ii) Find the coordinates of R. [5]