Quiz 3-A practice [20 marks]

1. [Maximum mark: 7] SPM.2.AHL.TZ0.8

The complex numbers w and z satisfy the equations

$$\frac{w}{z} = 2\mathrm{i}$$

$$\mathrm{z}^* - 3w = 5 + 5\mathrm{i}.$$

Find w and z in the form $a + b\mathrm{i}$ where $a, \mathrm{b} \in \mathbb{Z}$. [7]

Markscheme

substituting $w = 2\mathrm{i}z$ into $\mathrm{z}^* - 3w = 5 + 5\mathrm{i}$ ***M1***

$\mathrm{z}^* - 6\mathrm{i}z = 5 + 5\mathrm{i}$ ***A1***

let $z = x + y\mathrm{i}$

comparing real and imaginary parts of
$(x - y\mathrm{i}) - 6\mathrm{i}\,(x + y\mathrm{i}) = 5 + 5\mathrm{i}$ ***M1***

to obtain $x + 6y = 5$ and $-6x - y = 5$ ***A1***

attempting to solve for x and y) ***M1***

$x = -1$ and $y = 1$ so $z = -1 + \mathrm{i}$ ***A1***

hence $w = -2 - 2\mathrm{i}$ ***A1***

[7 marks]

2. [Maximum mark: 7] 25M.2.AHL.TZ1.8

Consider the functions $f,\ g$ and h defined as follows for $t\ \in \mathbb{R}$.

$$f(t) = \sin\,(2t + 1)$$

$$g(t) = \sin\,(2t + 3)$$

$h(t) = f(t) + g(t)$

(a) Show that $h(t) = \mathrm{Im}\left(\mathrm{e}^{2ti}\left(\mathrm{e}^{i} + \mathrm{e}^{3i}\right)\right)$. [2]

Markscheme

$\mathrm{Im}\ \left(\mathrm{e}^{2ti}\left(\mathrm{e}^{i} + \mathrm{e}^{3i}\right)\right)$

$= \mathrm{Im}\left(\mathrm{e}^{(2t+1)i} + \mathrm{e}^{(2t+3)i}\right)$ ***A1***

Note: This ***A1*** is for clearly showing that the powers are added.

Accept alternative notation for the step of adding the arguments e.g.

$(\cos\ (2t) + i\ \sin\ (2t))(\cos\ (1) + i\ \sin\ (1)) = \cos\ (2t+1) + i\ \sin\ (2t+1)$

$= \sin\ (2t+1) + \sin\ (2t+3)$ ***A1***

$= h(t)$ ***AG***

Note: Accept argument in reverse

[2 marks]

(b) Write $\mathrm{e}^{i} + \mathrm{e}^{3i}$ in the form $r\mathrm{e}^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [2]

Markscheme

$\mathrm{e}^{i} + \mathrm{e}^{3i} = 1.08060\ldots\mathrm{e}^{2i}$

$1.08\mathrm{e}^{2i}$ ***A1A1***

Note: Award ***A1*** for modulus ***A1*** for argument

$r = 1.08 (= 2 \cos(1))$, $\theta = 2$

[2 marks]

(c) Hence or otherwise, write $h(t)$ in the form $p \sin(2t + q)$, where $p > 0$ and $0 < q < 2\pi$. [3]

Markscheme

METHOD 1

attempt to use their answers to part (a) and (b) to write $h(t)$ as the imaginary part of a number in the form $re^{i\theta}$ ***(M1)***

$h(t) = \text{Im}\left(e^{2ti}\left(1.08060\ldots e^{2i}\right)\right)$

$= \text{Im}\left(1.08060\ldots e^{(2t+2)i}\right)$ ***(A1)***

$h(t) = 1.08060\ldots \sin(2t + 2)$

$= 1.08 \sin(2t + 2)\ (= 2\cos(1) \sin(2t + 2))$ ***A1***

METHOD 2

Considering the graph of $h(t)$

amplitude $= 1.08060\ldots$

$p = 1.08\ (= 2\cos(1))$ ***(A1)***

attempt to consider the horizontal shift of the graph of h ***(M1)***

first negative zero of graph is -1

$h(t) = 1.08060\ldots \sin(2(t + 1))\ (= 1.08060\ldots \sin(2t + 2))$

$= 1.08 \sin(2t + 2)\ (= 2\cos(1) \sin(2t + 2))$ ***A1***

[3 marks]

3. [Maximum mark: 6] 22M.1.AHL.TZ1.9

Consider the complex numbers $z_1 = 1 + b\mathrm{i}$ and $z_2 = \left(1 - b^2\right) - 2b\mathrm{i}$, where $b \in \mathbb{R},\ b \neq 0$.

(a) Find an expression for $z_1 z_2$ in terms of b. [3]

Markscheme

$z_1 z_2 = (1 + b\mathrm{i})\left(\left(1 - b^2\right) - (2b)\mathrm{i}\right)$

$= \left(1 - b^2 - 2\mathrm{i}^2 b^2\right) + \mathrm{i}\left(-2b + b - b^3\right)$ ***M1***

$= \left(1 + b^2\right) + \mathrm{i}\left(-b - b^3\right)$ ***A1A1***

Note: Award ***A1*** for $1 + b^2$ and A1 for $-b\mathrm{i} - b^3\mathrm{i}$.

[3 marks]

(b) Hence, given that $\arg(z_1 z_2) = \frac{\pi}{4}$, find the value of b. [3]

Markscheme

$\arg(z_1 z_2) = \arctan\left(\frac{-b-b^3}{1+b^2}\right) = \frac{\pi}{4}$ ***(M1)***

EITHER

$\arctan(-b) = \frac{\pi}{4}$ (since $1 + b^2 \neq 0$, for $b \in \mathbb{R}$) ***A1***

OR

$-b - b^3 = 1 + b^2$ (or equivalent) ***A1***

THEN

$b = -1$ ***A1***

[3 marks]