INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

Math AA

EXERCISES [Math-AA 1.9]

MATHEMATICAL INDUCTION

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O. Practice questions

1. [Maximum mark: 7] ***[with GDC]***

It is given that

$$1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4} \quad \text{(or otherwise } \sum_{r=1}^{n} r^3=\frac{n^2(n+1)^2}{4}\text{)}$$

(a) Show that the result is true

(i) for $n=1$,

(ii) for $n=2$,

(iii) for $n=3$.

(b) Find $1^3+2^3+3^3+\cdots+20^3$.

(c) Find the least value of n such that $1^3+2^3+3^3+\cdots+n^3>1000000$

2. [Maximum mark: 7] ***[with GDC]***

Show by mathematical induction that for any $n\in\mathbb{Z}^+$

$$1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4} \quad \text{(or otherwise } \sum_{r=1}^{n} r^3=\frac{n^2(n+1)^2}{4}\text{)}$$

Step 1: We **show** that the result is true for $n=1$

LHS =	RHS =
Hence LHS = RHS	

Step 2: We **assume** that the result is true for $n=k$ *[write down your assumption]*

Step 3: We **show** that the result is true for $n = k + 1$ *[write down your claim]*

[Present your proof]

[Write down the conclusion]
The result is true for $n = 1$. Assuming it is true for $n = k$, we proved it is also true for $n = k$. Hence, by mathematical induction, the result is true for any $n \in \mathbb{Z}^+$.

A-B. Exam style questions (SHORT OR LONG)

DIVISIBILITY

3. [Maximum mark: 7] ***[without GDC]***

Using mathematical induction, prove that $10^n - 1$ is a multiple of 9, for $n \in \mathbb{Z}^+$.

4. [Maximum mark: 7] ***[without GDC]***

Using mathematical induction, prove that $n^3 + 2n$ is divisible by 3, for $n \in \mathbb{Z}^+$.

5. [Maximum mark: 7] ***[without GDC]***

Using mathematical induction, prove that $5^n + 3$ is divisible by 4, for $n = 1, 2, \ldots$

6. [Maximum mark: 7] ***[without GDC]***

Using mathematical induction, prove that $2^{2n} - 3n - 1$ is divisible by 9, for $n = 1, 2, \ldots$

7. [Maximum mark: 7] ***[without GDC]***

Use mathematical induction to prove that $5^n + 9^n + 2$ is divisible by 4, for $n \in \mathbb{Z}^+$.

8. [Maximum mark: 10] ***[without GDC]***

Prove **by induction** that $12^n + 2(5^{n-1})$ is a multiple of 7 for $n \in \mathbb{Z}^+$

SERIES

9. [Maximum mark: 8] ***[without GDC]***

Use mathematical induction to prove that for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

10. [Maximum mark: 8] ***[without GDC]***

Use mathematical induction to prove that for $n \geq 2$

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n-1}{n!} = \frac{n!-1}{n!}$$

11. [Maximum mark: 7] ***[without GDC]***

Use mathematical induction to prove that for $n \in \mathbb{Z}^+$,

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

12. [Maximum mark: 7] ***[without GDC]***

Using mathematical induction, prove that for all positive integers

$$\sum_{r=1}^{n} (r+1)2^{r-1} = n(2^n)$$

13. [Maximum mark: 9] ***[without GDC]***

(a) Use mathematical induction to prove that

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}, n \in \mathbb{Z}^+.$$

[7]

(b) Hence find the sum $\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + \frac{1}{99}$ in the simplest form. [2]

14. [Maximum mark: 11] ***[with GDC]***

(a) Use mathematical induction to prove that

$(1)(1!) + (2)(2!) + (3)(3!) + \cdots + (n)(n!) = (n+1)! - 1$ where $n \in \mathbb{Z}^+$ [9]

(b) Find the minimum number of terms of the series for the sum to exceed 10^9. [2]

SEQUENCES

15. [Maximum mark: 7] ***[without GDC]***

An arithmetic sequence is defined recursively by

the first term u_1

and the recursive relation $u_{n+1} = u_n + d$

Prove by mathematical induction the general formula $u_n = u_1 + (n-1)d$, for $n \in \mathbb{Z}^+$

16. [Maximum mark: 7] ***[without GDC]***

A geometric sequence is defined recursively by

the first term u_1

and the recursive relation $u_{n+1} = u_n r$

Prove by mathematical induction the general formula $u_n = u_1 r^{n-1}$, for $n \in \mathbb{Z}^+$

17. [Maximum mark: 7] ***[without GDC]***

A sequence is defined recursively by

the first term $u_1 = 10$

and the recursive relation $u_{n+1} = 2u_n + 2$

Prove by mathematical induction that $u_n = 3(2)^{n+1} - 2$, for $n \in \mathbb{Z}^+$

18.** [Maximum mark: 9] ***[without GDC]***

A sequence is defined by the first two terms $u_1 = 5$, $u_2 = 8$, and the recursive relation

$$u_{n+1} = 2u_n - u_{n-1} .$$

Prove by induction that $u_n = 3n + 2$, for $n \in \mathbb{Z}^+$

INEQUALITIES

19*. [Maximum mark: 8] ***[without GDC]***

Prove by mathematical induction that

$$(n+1)! \geq 2^n n \qquad \text{for any integer } n \geq 3 .$$

20*. [Maximum mark: 8] ***[without GDC]***

Prove by mathematical induction that

$$3^n > n^2 + 2n \qquad \text{for any integer } n \geq 2 .$$

TRIGONOMETRY

21. [Maximum mark: 7] ***[without GDC]***

Prove by mathematical induction that for all $n \in Z^+, \sin x \neq 0$,

$$(\cos x)(\cos 2x)\cdots(\cos 2^{n-1}x) = \frac{\sin 2^n x}{2^n \sin x}$$

22.** [Maximum mark: 13] ***[without GDC]***

(a) Show that $\sin 2nx = \sin((2n+1)x)\ \cos x - \cos((2n+1)x)\ \sin x$. [2]

(b) **Hence** prove, by mathematical induction,

$$\cos x + \cos 3x + \cos 5x + \cdots + \cos((2n-1)x) = \frac{\sin 2nx}{2\sin x}$$

for all $n \in Z^+, \sin x \neq 0$ [8]

(c) Find the value of $\sin 5°\left(\cos 5° + \cos 15° + \cos 25°\right)$ [3]

23.** [Maximum mark: 10] ***[without GDC]***

(a) Show that $\cos 2nx = \cos((2n+1)x)\ \cos x + \sin((2n+1)x)\ \sin x$. [2]

(b) **Hence** prove, by mathematical induction,

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin((2n-1)x) = \frac{1-\cos 2nx}{2\sin x}$$

for all $n \in Z^+, \sin x \neq 0$ [8]

(c) Given that $\sin x + \sin 3x + \sin 5x + \cdots + \sin 99x = \frac{1-\cos ax}{2\sin x}$, find the value of a . [2]

DERIVATIVES

24. [Maximum mark: 7] ***[without GDC]***

Let $f(x) = \ln x$. Prove by mathematical induction that $f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$ for $n \geq 1$

25. [Maximum mark: 10] ***[without GDC]***

Let $f(x) = e^{3x}$

(a) Find the first three derivatives of $f(x)$, and thus guess a formula for $f^{(n)}(x)$. [4]

(b) Prove by mathematical induction that your guess is true for all $n \in \mathbb{Z}^+$. [6]

26. [Maximum mark: 15] ***[without GDC]***

The function f is defined by $f(x)=\ln(x+1)$

Let $f^{(n)}(x)$ denote the result of differentiating $f(x)$ with respect to x, n times.

(a) Find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$, each in the form $\dfrac{a}{(x+1)^m}$ [5]

(b) Guess a formula for $f^{(n)}(x)$. [3]

(c) Use mathematical induction to prove that your guess is true for all $n\in\mathbb{Z}^+$. [7]

27. [Maximum mark: 10] ***[without GDC]***

Given that the derivative of x is 1, and using only the product rule, prove by mathematical induction that $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$, for all positive integer values of n.

28. [Maximum mark: 7] ***[without GDC]***

Using mathematical induction, prove that $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}(\cos x)=\cos\left(x+\dfrac{n\pi}{2}\right)$ for all positive integer values of n.

29. [Maximum mark: 7] ***[without GDC]***

The function f is defined by $f(x)=\mathrm{e}^{px}(x+1)$, here $p\in\mathbb{R}$.

(i) Show that $f'(x)=\mathrm{e}^{px}(p(x+1)+1)$.

(ii) Let $f^{(n)}(x)$ denote the result of differentiating $f(x)$ with respect to x, n times.

Use mathematical induction to prove that

$$f^{(n)}(x)=p^{n-1}\mathrm{e}^{px}(p(x+1)+n),\quad n\in\mathbb{Z}^+$$

30. [Maximum mark: 12] ***[without GDC]***

Given that $y=x\mathrm{e}^{-x}$,

(a) find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(b) use mathematical induction to prove that, for $n\in\mathbb{Z}^+$, $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}=(-1)^{n+1}\mathrm{e}^{-x}(n-x)$

31. [Maximum mark: 9] ***[without GDC]***

The function f is defined by $f(x)=x\mathrm{e}^{2x}$. Use mathematical induction to prove that

$$f^{(n)}(x)=(2^n x+n2^{n-1})\mathrm{e}^{2x}$$

for all $n\in\mathbb{Z}^+$, where $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.

32. [Maximum mark: 8] ***[without GDC]***

The function f is defined by $f(x) = \sin 2x$. Use mathematical induction to prove that

$$f^{(2n)}(x) = (-4)^n \sin 2x$$

for all $n \in \mathbb{Z}^+$, where $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.

COMPLEX NUMBERS

33. [Maximum mark: 10] ***[without GDC]***

(a) Evaluate $(1 + \mathrm{i})^2$, where i = $\sqrt{-1}$. [2]

(b) Prove, by mathematical induction, that $(1 + \mathrm{i})^{4n} = (-4)^n$, where $n \in \mathbb{N}^*$. [6]

(c) Hence or otherwise, find $(1 + \mathrm{i})^{32}$. [2]

34. [Maximum mark: 7] ***[without GDC]***

Use mathematical induction to prove De Moivre's theorem

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos(n\theta) + \mathrm{i}\sin(n\theta), \qquad n \in \mathbb{Z}^+.$$

35. [Maximum mark: 7] ***[without GDC]***

Use mathematical induction to prove that for $n \in \mathbb{Z}^+$,

$$\left(1 + \sqrt{3}\,\mathrm{i}\right)^n = 2^n\left(\cos\frac{n\pi}{3} + \mathrm{i}\sin\frac{n\pi}{3}\right)$$