INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

Math AA

EXERCISES [Math-AA 1.8]

METHODS OF PROOF

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O. Practice questions

1. [Maximum mark: 6] ***[without GDC]***

Let $a \in \mathbb{Z}$. When a is divided by 3 it leaves a remainder 0, 1 or 2; thus a has one of the forms $3n$, $3n+1$ or $3n+2$.

(a) Prove the statement

"if a is a multiple of 3, then a^2 is also a multiple of 3" [2]

(b) Prove by **contradiction** the converse statement, i.e.

"if a^2 is a multiple of 3, then a is also a multiple of 3" [4]

2. [Maximum mark: 4] ***[without GDC]***

(a) Prove the statement

"if a is a multiple of 8, then a^2 is also a multiple of 8". [2]

(b) Provide a **counterexample** to show that the converse statement is not true, i.e.

"if a^2 is a multiple of 8, then a is not necessarily a multiple of 8" [2]

3. [Maximum mark: 5] ***[without GDC]***

(a) Expand $(5n+r)^2$. [1]

(b) Hence, prove by **contradiction**, that

"if a^2 is a multiple of 5, then a is also a multiple of 5" [4]

4. [Maximum mark: 6] ***[without GDC]***

It is known that for an integer a, if a^2 is a multiple of 3 then a is also a multiple of 3. Use this fact to show that $\sqrt{3}$ is irrational, by using **contradiction**.

5. [Maximum mark: 6] ***[without GDC]***

It is known that $\sqrt{2}$ is irrational. Prove that $\sqrt{2}+1$ is also irrational.

6. [Maximum mark: 8] ***[without GDC]***

Prove the following statements by using **contradiction**

(a) There exist no integers a and b for which $6a+2b=35$ [2]

(b) There exist no integers a and b for which $6a+9b=35$ [2]

(c) The equation $2x^5-6x^2+14x=1$ has no integer solution. [2]

(d) The equation $3x^5+15x^2-14=0$ has no integer solution. [2]

7. [Maximum mark: 14] ***[without GDC]***

Rational numbers, by definition, have the form of a fraction $\frac{a}{b}$ where $a \in Z$ and $b \in Z^*$.

(a) Prove that the sum and the product of two rational numbers x and y are also rational numbers. [4]

(b) Peter claims that the sum and the product of two irrational numbers x and y are also irrational. Use **counterexamples** to prove that both Peter's claims are false. [4]

(c) Prove by **contradiction** that the **sum** of a rational number x and an irrational number y is irrational. [3]

(d) Prove by **contradiction** that the **product** of a non-zero rational number x and an irrational number y is irrational. [3]

A. Exam style questions (SHORT)

8. [Maximum mark: 7] ***[without GDC]***

(a) Prove by contradiction that for an integer a: "if a^3 is even, then a is even" [3]

(b) Prove by contradiction that $\sqrt[3]{2}$ is irrational. [4]

9. [Maximum mark: 7] ***[without GDC]***

Given that π is irrational, show that $2\pi+1$ is also irrational.

10. [Maximum mark: 7] ***[without GDC]***

The number $\log_a b$ is defined as follows

$$\log_a b = x \text{ if and only if } a^x = b$$

Show that the number $\log_2 3$ is irrational.

11. [Maximum mark: 4] ***[without GDC]***

Prove by using **contradiction** that the cubic polynomial

$$p(x) = 2x^3 + 8x^2 - 6x + 1$$

has no integer zeros.

12. [Maximum mark: 5] ***[without GDC]***

Prove by using **contradiction** that the cubic polynomial

$$p(x) = 2x^3 + x^2 + x - 3$$

has no integer zeros.

13. [Maximum mark: 8] ***[without GDC]***

(a) Prove the identity $2x^3 + 7x^2 - 14x + 5 \equiv (x+5)(2x^2 - 3x + 1)$ [2]

(b) Given that $3x^3 + 13x^2 - 3x + 35 \equiv (x+5)(ax^2 + bx + c)$ find the values of a, b, c. [3]

(c) Given that $ax^3 + bx^2 - 23x + c \equiv (x+5)(2x^2 + dx + 2)$ find the values of a, b, c, d. [3]

14. [Maximum mark: 6] ***[without GDC]***

Let $n \in \mathbb{Z}$. Prove the following statements:

(a) If n is an odd integer then $n^2 + 2n + 5$ is even. [4]

(b) If $n^2 + 2n + 5$ is odd then n is even. [2]

15. [Maximum mark: 4] ***[without GDC]***

(a) For any $x, y, z \in R$, prove that $x^2 + y^2 + z^2 = 0 \Rightarrow xyz = 0$ [2]

(b) Is the converse of (a) true? Justify your answer. [2]

16. [Maximum mark: 6] ***[without GDC]***

Suppose $a, b, c \in Z$. Prove by contradiction that

(a) If $a + b \geq 19$, then $a \geq 10$ or $b \geq 10$. [3]

(b) If $a + b + c \geq 19$, then $a \geq 7$ or $b \geq 7$ or $c \geq 7$. [3]

17. [Maximum mark: 4] ***[without GDC]***

Suppose a, b are positive real numbers. Prove the statement

If $ab = c$ then $a \leq \sqrt{c}$ or $b \leq \sqrt{c}$.

18. [Maximum mark: 4] ***[without GDC]***

Suppose $a, b \in Z$. Given that $5a + 3b \geq 81$, prove that $a \geq 11$ or $b \geq 11$.

19*. [Maximum mark: 6] ***[without GDC]***

(a) Suppose that $\sum_{i=1}^{9} x_i \geq 46$, where all x_i are integers. Prove by contradiction that at least one of the values x_i is greater than or equal to 6. [3]

(b) Suppose that $\sum_{i=1}^{9} x_i > 45$, where all x_i are real numbers. Prove by contradiction that at least one of the values x_i is greater than 5. [3]

20*. [Maximum mark: 5] ***[without GDC]***

Prove that $n^3 - n$ is a multiple of 6.

21. [Maximum mark: 4] ***[without GDC]***

Peter claims that the product of three consecutive integers greater than 1 is always a multiple of 12. Prove that Peter's claim is false. State the method of proof.

B. Exam style questions (LONG)

22. [Maximum mark: 10] ***[without GDC]***

Integers divided by 3 leave a remainder 0, 1 or 2; thus, any integer has one of the forms

$3n$ (multiples of 3)

$3n+1$ (leave remainder 1)

$3n+2$ (leave remainder 2)

(a) Prove that the sum of two multiples of 3 is a multiple of 3. [2]

(b) For two numbers that are not multiples of 3

Ann claims that their sum is always a multiple of 3

Bill claims that their sum is always a non-multiple of 3

Show that both claims are wrong. [3]

(c) Prove that for any integer a, the number a^2 cannot be of the form $3n+2$. [5]

23. [Maximum mark: 12] ***[without GDC]***

Consider the equation of integers $a+b+c=d$.

(a) Use a deductive proof to prove the statements

(i) "if a,b,c are all even then d is also even";

(ii) "if a,b,c are all odd then d is also odd" [5]

(b) Use a **counterexample** to disprove the statement

"if d is odd then a,b,c are all odd". [2]

(c) State whether the following statement is true or false and prove your claim.

"if d is even then a,b,c are all even". [3]

(d) Use **contradiction** to prove the statement

"if d is even then at least one of a,b,c is even". [2]

24*. [Maximum mark: 12] ***[without GDC]***

(a) Prove the identity $a^5-1\equiv(a-1)(a^4+a^3+a^2+a+1)$ [2]

(b) By letting $a=\dfrac{x}{y}$ deduce an identity for x^5-y^5 with no fractions. [3]

(c) Write down similar identities for a^n-1 and x^n-y^n, $n\in Z^+$ [4]

(d) Replace y with $-y$ in the last identity to deduce a similar result for x^n+y^n,

where n is odd. [3]

25*. [Maximum mark: 10] ***[without GDC]***

Let $f(x)=2x+3$, $g(x)=\dfrac{x+3}{2x-4}$ and $h(x)=x^2+1$

(a) Show that $f(x_1)=f(x_2)\Rightarrow x_1=x_2$ [2]

(b) Show that $g(x_1)=g(x_2)\Rightarrow x_1=x_2$ [4]

(c) Show that $x_1\neq x_2\Rightarrow g(x_1)\neq g(x_2)$ [2]

(d) Show that $x_1\neq x_2 \not\Rightarrow h(x_1)\neq h(x_2)$ by using a counterexample. [2]

26*. [Maximum mark: 10] ***[without GDC]***

In a birthday party there are **50 people** who may give handshakes to each other.

(a) Show, by using contradiction, that there are at least two people who give the same number of handshakes. [5]

Each person is asked to write on a piece of paper their favorite day of the week.

(b) Show that at least one of the days will appear at least 8 times. [5]