INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

Math-AA

EXERCISES [Math-AA 2.11-2.12]

POLYNOMIALS

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O. Practice questions

1. [Maximum mark: 6] ***[with GDC]***

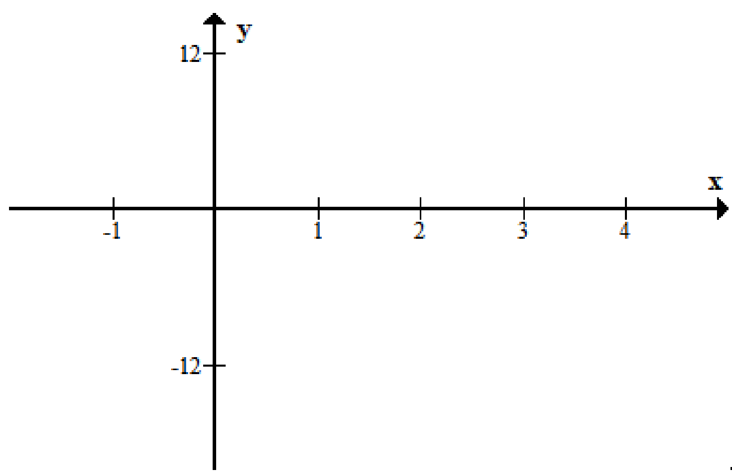
The cubic polynomial $f(x) = 2x^3 - 12x^2 + 22x - 12$ has **three** real zeros

(a) Show that $x = 1$ is a zero of $f(x)$. [1]

(b) Find the remaining 2 zeros of $f(x)$ [using your GDC] [2]

(c) Hence, express $f(x)$ in the form $f(x) = a(x-p)(x-q)(x-r)$ [1]

(d) Sketch the graph of $y = f(x)$; indicate $x-$intercepts and $y-$intercepts. [2]



2. [Maximum mark: 6] ***[without GDC]***

Consider the cubic polynomial $f(x) = 2x^3 - 12x^2 + 22x - 12$.

It is given that $x = 1$ is a zero of $f(x)$, that is $f(1) = 0$.

(a) Use long division by $(x-1)$ to express $f(x)$ in the form $f(x) = (x-1)q(x)$, where $q(x)$ is a quadratic. [3]

(b) Hence, find the remaining zeros of $f(x)$. [3]

(c) Use long division of $f(x)$ by $q(x)$ to reconfirm the result found in (a). [2]

3. [Maximum mark: 6] ***[without GDC]***

(a) Use long division of $f(x)=2x^3+9x^2+16x+5$

(i) by x^2+3x+1 to express $f(x)$ in the form $f(x)=(x^2+3x+1)q(x)+r(x)$.

(i) by $x-2$ to express $f(x)$ in the form $f(x)=(x-2)Q(x)+R$.

(b) **Hence**, express

(i) $\dfrac{2x^3+9x^2+16x+5}{x^2+3x+1}$ in the form $ax+b+\dfrac{cx+d}{x^2+3x+1}$

(ii) $\dfrac{2x^3+9x^2+16x+5}{x-2}$ in the form $A(x)+\dfrac{B}{x-2}$

4. [Maximum mark: 4] ***[without GDC]***

Let $f(x)=6x^2+3x+2$ and $g(x)=2x-5$

(a) Use long division to express $f(x)$ in the form $f(x)=g(x)q(x)+r$.

(b) Expand $g(x)q(x)+r$ found in (a) to confirm the result.

(c) **Hence**, express $h(x)=\dfrac{6x^2-3x+2}{2x-5}$ in the form $Ax+B+\dfrac{C}{2x-5}$

5. [Maximum mark: 4] ***[without GDC]***

Use long division to express $f(x)=\dfrac{5x^2-3x+2}{x^2-3x-1}$ in the form $A+\dfrac{Bx+C}{x^2-3x-1}$

6. [Maximum mark: 8] ***[without GDC]***

Consider the cubic function $f(x)=ax^3+2x^2+3x+4$. Find the value of a in each of the following cases

(a) the graph of the function passes through the point (1,10). [2]

(b) $f(x)$ is divisible by $(x-1)$. [2]

(c) when $f(x)$ is divided by $(x-1)$, the remainder is 10. [2]

(d) Confirm the result in (c) by using long division. [2]

7. [Maximum mark: 5] ***[without GDC]***

Let $f(x)=x^5+x^4+x^3+x^2+x+1$.

(a) Use the **factor theorem** to show that $f(x)$ is divisible by $x+1$ [2]

(b) Use the **remainder theorem** to find

(i) the remainder when $f(x)$ is divided by $x-1$.

(i) the remainder when $f(x)$ is divided by $x-2$. [3]

8. [Maximum mark: 5] ***[without GDC]***

The polynomial $f(x)=x^5+x^4+x^3+x^2+ax+b$ is divisible by $x+1$ and leaves a remainder of 6 when divided by $x-1$. Find the values of a and b

9. [Maximum mark: 13] ***[without GDC]***

$f(x) = ax^3 + bx^2 + 4x + 4$ is divisible by $x^2 - 3x + 2$. Find the values of a and b.

10*. [Maximum mark: 13] ***[without GDC]***

When $f(x) = ax^3 + bx^2 + 3x + 4$ is divided by $x^2 - 3x + 2$ the remainder is $2x + 2$

Find the values of a and b

11. [Maximum mark: 10] ***[with / without GDC]***

Consider the cubic function $f(x) = 2x^3 + ax^2 + bx + c$. Find the values of a, b, c

(a) if the graph of the function passes through the points (1,0), (–1 ,2), and (0,3). [5]

(b) if the graph of the function passes through the points (1,0), (–1,0), and (3,0). [5]

12. [Maximum mark: 8] ***[without GDC]***

Consider the polynomial $f(x) = 4x^3 - 4x^2 - 5x + 3$.

(a) Show that $x = -1$ is a root of $f(x)$. [1]

(b) Find the other two roots of $f(x)$ by using long division. [5]

(c) Express the polynomial in the form $f(x) = (x - a)(bx - c)(dx - e)$, where $a, b, c, d, e \in Z$. [2]

13. [Maximum mark: 12] ***[with GDC]***

For the general form of a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$

Sum of roots $S = -\dfrac{a_{n-1}}{a_n}$ Product of roots $P = (-1)^n \dfrac{a_0}{a_n}$

Use your GDC to find the roots of the following polynomials and then confirm the formulas for S and P (as in the first example).

Roots	Their sum	Their product	S	P
$f(x) = 2x^3 - 12x^2 + 22x - 12$				
1, 2, 3	$1+2+3=6$	$1\times2\times3=6$	$S = -\dfrac{-12}{2} = 6$	$P = -\dfrac{-12}{2} = 6$
$f(x) = 4x^3 - 16x^2 + 19x - 6$				
$f(x) = 4x^4 - 20x^3 + 35x^2 - 25x + 6$				
$f(x) = x^3 + 6x^2 + 9x$				

14. [Maximum mark: 12] ***[without GDC]***

Complete the following table: for each polynomial find the sum and the product of the roots (allowing non-real roots and repetition of roots).

Polynomial	Sum of roots	Product of roots
$f(x) = 2x^4 + 6x^3 + 5x^2 - 7x + 8$		
$f(x) = 2x^5 + 6x^3 + 5x^2 - 7x + 8$		
$f(x) = x^{10} - x^9 - 1$		

15. [Maximum mark: 4] ***[without GDC]***

The cubic function $f(x) = ax^3 + 6x^2 - 8x - 9$ has three real roots. Their sum is -2.

(a) Find the value of a. [2]

(b) Find the product of the three roots. [2]

16. [Maximum mark: 4] ***[with GDC]***

The cubic $f(x) = ax^3 + bx^2 - 9x - 8$ has three real roots. The sum of the roots is -3 while their product is 4. Find the three roots.

17. [Maximum mark: 12] ***[without GDC]***

Let α, β be the (non-real) roots of the quadratic $y = 2x^2 + 4x + 6$

(a) Write down the values of (i) $\alpha + \beta$ (ii) $\alpha\beta$. [2]

(b) Find the value of $\alpha^2 + \beta^2$. [2]

(c) Find the value of $(\alpha - \beta)^2$. [2]

(d) Find a quadratic with roots $\alpha^2\beta$ and $\alpha\beta^2$ [3]

(e) Find a quadratic with roots α^2 and β^2 [3]

A. Exam style questions (SHORT)

18. [Maximum mark: 4] ***[without GDC]***

Consider $f(x) = x^3 - 2x^2 - 5x + k$. Find the value of k if $(x+2)$ is a factor of $f(x)$.

19. [Maximum mark: 4] ***[without GDC]***

When the function $f(x) = 6x^4 + 11x^3 - 22x^2 + ax + 6$ is divided by $(x+1)$ the remainder is –20. Find the value of a.

20. [Maximum mark: 4] ***[without GDC]***

When $x^4 + ax + 3$ is divided by $(x-1)$, the remainder is 8. Find the value of a.

21. [Maximum mark: 10] ***[without GDC]***

The polynomial $p(x) = x^3 + ax^2 - 3x + b$ is divisible by $(x-2)$ and has a remainder 6 when divided by $(x+1)$.

(a) Find the value of a and of b. [5]

(b) Factorise completely $p(x)$ and state its roots. [5]

22*. [Maximum mark: 6] ***[without GDC]***

The polynomial $p(x) = (ax+b)^3$ leaves a remainder of –1 when divided by $(x+1)$, and a remainder of 27 when divided by $(x-2)$. Find the values of a and b.

23. [Maximum mark: 6] ***[without GDC]***

The polynomial $f(x) = x^3 + 3x^2 + ax + b$ leaves the same remainder when divided by $(x-2)$ as when divided by $(x+1)$.

(a) Find the value of a. [5]

(b) State the possible values of b. [1]

24. [Maximum mark: 7] ***[without GDC]***

Given that $(x-2)$ and $(x+2)$ are factors of $f(x) = x^3 + px^2 + qx + 4$,

(a) find the value of p and of q. [5]

(b) solve the equation $f(x) = 0$. [2]

25. [Maximum mark: 7] ***[without GDC]***

The polynomial $P(x)=2x^3+ax^2-4x+b$ is divisible by $(x-1)$ and by $(x+3)$.

(a) Find the value of a and of *b*. [5]

(b) Factorise $f(x)$. [2]

26. [Maximum mark: 8] ***[without GDC]***

The polynomial x^2-4x+3 is a factor of $x^3+(a-4)x^2+(3-4a)x+3$.

(a) Calculate the value of the constant a. [4]

(b) Factorise completely the cubic polynomial. [2]

(c) Find the remainder when the cubic is divided by $(x-2)$. [2]

27*. [Maximum mark: 13] ***[without GDC]***

Consider the cubic function $f(x)=ax^3+bx^2+3x+4$. Find the values of a and b in each of the following cases

(a) $f(x)$ is divisible by $(x-1)$ and leaves a remainder 6 when divided by $(x+1)$. [4]

(b) $f(x)$ is divisible by (x^2-1). [4]

(c) $f(x)$ leaves a remainder $-3x+3$ when divided by (x^2-1). [5]

28. [Maximum mark: 6] ***[with GDC]***

When $P(x)=4x^3+px^2+qx+1$ is divided by $(x-1)$ the remainder is –2. When $P(x)$ is divided by $(2x-1)$ the remainder is $\frac{13}{4}$. Find the value of p and of q.

29*. [Maximum mark: 7] ***[with GDC]***

When $P(x)=4x^3+px^2+qx+1$ is divided by $(x-1)(2x-1)$ the remainder is $\frac{17-21x}{2}$.

Find the value of p and of q.

30. [Maximum mark: 7] ***[without GDC]***

Let α,β be the roots of the quadratic $f(x)=x^2-2x+5$. Without finding α and β

(a) Find the value of $\alpha^2+\beta^2$ [3]

(b) Find a quadratic with integer coefficients and roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$. [4]

31. [Maximum mark: 6] ***[without GDC]***

The roots α and β of the quadratic equation $x^2-kx+(k+1)=0$ are such that $\alpha^2+\beta^2=13$. Find the possible values of the real number k.

B. Exam style questions (LONG)

32. [Maximum mark: 12] ***[without GDC]***

The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ is divisible by $x^2 - 3x + 2$.

The sum of its roots is 7 and the product of its roots is 0.

(a) Find the values of a, b, c and d. [8]

(b) Factorize $f(x)$. [4]

33. [Maximum mark: 10] ***[without GDC]***

The polynomial $f(x) = x^4 - 2x^3 + ax^2 + bx + 3$ is divisible by $(x-1)$ and the quotient of the division is the polynomial $q(x)$.

(a) Find the sum and the product of the roots of $f(x)$. [2]

(b) State the degree of $q(x)$. [1]

(c) Find the sum of a and b. [3]

(d) Find the sum and the product of the roots of $q(x)$. [4]

34. [Maximum mark: 12] ***[without GDC]***

Let $f(x) = ax^4 + bx^3 + cx^2 + dx + 16$. The sum and the product of the roots of $f(x)$ are both 8.

(a) Find the values of a and b. [3]

The polynomial $f(x)$ is divisible by $(x-1)$ and $f(x) = (x-1)q(x)$.

When $f(x)$ is divided by $(x+1)$ the remainder is 120.

(b) Find the values of c and d. [5]

(c) Find the sum and the product of the roots of $q(x)$. [4]

35*. [Maximum mark: 20] ***[with GDC]***

Let α, β be the roots of the quadratic $f(x) = 5x^2 - 2x - 4$. Without finding α and β

(a) Write down the values of (i) $\alpha + \beta$ (ii) $\alpha\beta$. [2]

(b) Find the values of (i) $\alpha^2 + \beta^2$ (ii) $\alpha^3 + \beta^3$ [5]

(c) Find a quadratic with integer coefficients which has roots $\frac{1}{\alpha}, \frac{1}{\beta}$. [5]

(d) Find a quadratic with integer coefficients and roots α^2 and β^2 [4]

(e) Find a quadratic with integer coefficients and roots α^3 and β^3 [4]

36*. [Maximum mark: 16] ***[without GDC]***

For a cubic function $ax^3 + bx^2 + cx + d$ with roots r_1, r_2, r_3, it is given that

$$S_1 = r_1 + r_2 + r_3 = -\frac{b}{a}, \quad S_2 = r_1r_2 + r_2r_3 + r_3r_1 = \frac{c}{a}, \quad S_3 = r_1r_2r_3 = -\frac{d}{a}$$

Let α, β, γ be the roots of the cubic function $f(x) = x^3 - 5x^2 - 7x + 3$

Without evaluating the roots α, β, γ, find

(a) $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. [3]

(b) $\alpha^2 + \beta^2 + \gamma^2$. [4]

(c) $(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2$. [5]

(d) a cubic polynomial which has roots $\alpha^2, \beta^2, \gamma^2$. [4]

37*. [Maximum mark: 15] ***[with GDC]***

Consider the following cubic polynomial

$$f(x) = (x-1)(x^2 + (2-k)x + k^2)$$

(a) Show that $x = 1$ cannot be a root of the quadratic factor $(x^2 + (2-k)x + k^2)$. [4]

(b) Find the values of k in each of the following cases

(i) if the polynomial has exactly one real root;

(ii) if the polynomial has exactly two distinct real roots;

(iii) if the polynomial has three distinct real roots. [7]

(c) Find the roots of $f(x)$ for each value of k in case (b)(ii). [4]