INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

Math AA

EXERCISES [Math-AA 1.13-1.15]

COMPLEX NUMBERS (POLAR FORM)

Compiled by Christos Nikolaidis

POLAR FORM – DE MOIVRE

O. Practice questions

1. [Maximum mark: 16] ***[with / without GDC]***

Write down the polar form of the following complex numbers

Cartesian form	**Polar form**		
$x+yi$	$r\,\text{cis}\,\theta$	$r(\cos\theta+\text{i}\sin\theta)$	$r\,\text{e}^{\text{i}\theta}$
1			
-1			
i			
$-\text{i}$			
8			
-8			
8i			
-8i			

2. [Maximum mark: 4] ***[with / without GDC]***

Write down the Cartesian form of the following complex numbers

Polar form	**Cartesian form**
$4\text{cis}0$	
$5\text{cis}\pi$	

Polar form	**Cartesian form**
$6\text{cis}\dfrac{\pi}{2}$	
$7\text{cis}\left(-\dfrac{\pi}{2}\right)$	

3. [Maximum mark: 8] ***[without GDC]***

Let $z = 5 + 5\sqrt{3}\mathrm{i}$ and $z^* = 5 - 5\sqrt{3}\mathrm{i}$

(a) Find the modulus and the argument of z [4]

(b) Find the modulus and the argument of z^* [2]

(c) Write down their polar form in 3 different ways (complete the table below). [2]

	$r\mathrm{cis}\theta$	$r(\cos\theta + \mathrm{i}\sin\theta)$	$r\mathrm{e}^{\mathrm{i}\theta}$
z			
z^*			

4. [Maximum mark: 4] ***[with GDC]***

Let $z = 3 + 4\mathrm{i}$

(a) Find the modulus and the argument of z [2]

(b) Write down its polar form of z in 3 different ways (complete the table below). [2]

	$r\mathrm{cis}\theta$	$r(\cos\theta + \mathrm{i}\sin\theta)$	$r\mathrm{e}^{\mathrm{i}\theta}$
z			

5. [Maximum mark: 10] ***[with / without GDC]***

Find the polar form $r\mathrm{cis}\theta$ of the complex numbers

$$z_1 = 1 + \mathrm{i}, \qquad z_2 = -1 + \mathrm{i}, \qquad z_3 = -1 - \mathrm{i}, \qquad z_4 = 1 - \mathrm{i}$$

and represent them in the complex plane.

6. [Maximum mark: 16] ***[without GDC]***

Let $z = 3\mathrm{cis}\,0.3 = 3\mathrm{e}^{0.3\mathrm{i}}$ and $w = 2\mathrm{cis}\,0.2 = 2\mathrm{e}^{0.2\mathrm{i}}$

(a) Complete the following tables [10]

$z = 3\,\mathrm{cis}\,0.3$ $w = 2\,\mathrm{cis}\,0.2$ find in the same form	
$z\,w$	
$\frac{z}{w}$	
z^2	
z^3	
z^4	

$z = 3\mathrm{e}^{0.3\mathrm{i}}$ $w = 2\mathrm{e}^{0.2\mathrm{i}}$ find in the same form	
$z\,w$	
$\frac{z}{w}$	
z^2	
z^3	
z^4	

(b) Find z^2w^3 (i) in the form $r\,\text{cis}\,\theta$ (ii) in the form $r\,\text{e}^{\text{i}\theta}$ [4]

(c) Find $\dfrac{z^2}{w^3}$ [2]

7. [Maximum mark: 15] ***[without GDC]***

Let $z = 2\text{cis}10°$ and $w = 6\text{cis}20°$

(a) Express the following results in the polar form $r\text{cis}\theta$

zw	
$\dfrac{w}{z}$	
$\dfrac{z}{w}$	
z^2	
z^3	
$\dfrac{z^3}{w^2}$	

[12]

(b) Express $zw + z^3$ in polar and **hence** in Cartesian form $x + y\text{i}$. [3]

8. [Maximum mark: 6] ***[without GDC]***

Let $z = 2\text{cis}10°$. Find the minimum positive integer n such that z^n is

(i) a real number (ii) a positive real number (iii) an imaginary number

9. [Maximum mark: 14] ***[without GDC]***

(a) Find the polar form of the complex numbers $z = 1 + \text{i}$ and $w = \sqrt{3} - \text{i}$ using degrees for the arguments. [4]

(b) Find zw by using the Cartesian forms. [2]

(c) Find zw by using the polar forms. [2]

(d) Deduce the exact values of $\cos 15°$, $\sin 15°$ and $\tan 15°$. [6]

A. Exam style questions (SHORT)

10. [Maximum mark: 10] ***[without GDC]***

Let $z=\sqrt{3}+\mathrm{i}$

(a) Find the polar form $r\mathrm{cis}\theta$ of the complex numbers z and $-z$. [4]

(b) Find the polar form $r\mathrm{cis}\theta$ of the complex numbers $\overline{z}$ and $-\overline{z}$. [4]

(c) Represent the four complex numbers above in the Complex plane. [2]

11. [Maximum mark: 9] ***[without GDC]***

Find the polar form of the complex numbers $z=1+\mathrm{i}\sqrt{3}$, $w=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\mathrm{i}$, $u=-2\sqrt{3}+2\mathrm{i}$

12. [Maximum mark: 8] ***[without GDC]***

Given that $z=2\,\mathrm{cis}\frac{\pi}{3}$, $w=\mathrm{cis}(-\frac{\pi}{4})$, $u=4\mathrm{cis}\frac{5\pi}{6}$, find the polar form of

(i) zw (ii) $\frac{z}{w}$ (iii) u^3 (iv) $\frac{z^6}{w^4u^3}$

13. [Maximum mark: 5] ***[without GDC]***

(a) Find the polar form of $z=1-\mathrm{i}\sqrt{3}$

(b) Hence, express $\frac{1}{(1-\mathrm{i}\sqrt{3})^3}$ in the form $\frac{a}{b}$ where $a,b\in Z$.

14. [Maximum mark: 6] ***[without GDC]***

Let $z_1=a\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)$ and $z_2=b\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$

Express $\left(\frac{z_1}{z_2}\right)^3$ in the form $z=x+y\mathrm{i}$.

15. [Maximum mark: 8] ***[without GDC]***

Let $z_1=r\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)$ and $z_2=1+\sqrt{3}\,\mathrm{i}$.

(a) Write z_2 in modulus-argument form. [3]

(b) Find the value of r if $\left|z_1{z_2}^3\right|=2$ [3]

(c) Find the argument of $z_1{z_2}^3$. [2]

16. [Maximum mark: 12] ***[without GDC]***

Let $z_1 = \frac{\sqrt{6} - \mathrm{i}\sqrt{2}}{2}$, and $z_2 = 1 - \mathrm{i}$.

(a) Write z_1 and z_2 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. [6]

(b) Show that $\frac{z_1}{z_2} = \cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}$ [2]

(c) Find the value of $\frac{z_1}{z_2}$ in the form $a + b\mathrm{i}$, where a and b are to be determined exactly in radical (surd) form. Hence or otherwise find the exact values of $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$. [4]

17. [Maximum mark: 9] ***[without GDC]***

Consider $z = \cos\theta + \mathrm{i}\sin\theta$

(a) Find z^3 by using de Moivre's theorem [1]

(b) Find z^3 by using the binomial theorem [3]

(c) Compare the results for z^3 and deduce formulas for $\cos 3\theta$ and $\sin 3\theta$; express $\cos 3\theta$ in terms of $\cos\theta$ only and $\sin 3\theta$ in terms of $\sin\theta$ only. [5]

18. [Maximum mark: 9] ***[without GDC]***

Consider $z = \cos\theta + \mathrm{i}\sin\theta$

(a) Find z^4 by using de Moivre's theorem. [1]

(b) Find z^4 by using the binomial theorem. [4]

(c) Compare the results for z^4 and deduce formulas for $\cos 4\theta$ and $\sin 4\theta$. [4]

19. [Maximum mark: 16] ***[without GDC]***

Consider $z = \cos\theta + \mathrm{i}\sin\theta$

(a) Use de Moivre's theorem to show that

$$z^n + z^{-n} = 2\cos n\theta$$
$$z^n - z^{-n} = 2\mathrm{i}\sin n\theta.$$ [4]

(b) Expand $(z + z^{-1})^2$ and $(z - z^{-1})^2$ and deduce formulas for $\cos^2\theta$ and $\sin^2\theta$. [6]

(c) Expand $(z + z^{-1})^3$ and $(z - z^{-1})^3$ and deduce formulas for $\cos^3\theta$ and $\sin^3\theta$. [6]

20. [Maximum mark: 12] ***[without GDC]***

Consider $z = \cos\theta + \mathrm{i}\sin\theta$

(a) Use de Moivre's theorem to show that

$$z^n + z^{-n} = 2\cos n\theta$$
$$z^n - z^{-n} = 2\mathrm{i}\sin n\theta.$$ [4]

(b) Expand $(z + z^{-1})^4$ and $(z - z^{-1})^4$ and deduce formulas for $\cos^4\theta$ and $\sin^4\theta$. [8]

21. [Maximum mark: 6] ***[without GDC]***

Given that $z=(b+\mathrm{i})^2$, where b is real and positive, find the **exact** value of b when $\arg z = 60°$.

METHOD A: Expand $(b+\mathrm{i})^2$ and then consider the argument of z
METHOD B: Think of the argument of $b+\mathrm{i}$ first

22*. [Maximum mark: 7] ***[without GDC]***

Find, in its simplest form, the argument of $(\sin\theta+\mathrm{i}(1-\cos\theta))^2$ where θ is an acute angle.

23*. [Maximum mark: 6] ***[without GDC]***

The complex numbers z_1 and z_2 are $z_1=2+\mathrm{i}$, $z_2=3+\mathrm{i}$.

(a) Find z_1z_2, giving your answer in the form $a+\mathrm{i}b$, $a,b\in R$. [1]

(b) The polar form of z_1 may be written as $\left(\sqrt{5},\arctan\frac{1}{2}\right)$.

(i) Express the polar form of z_2, z_1z_2 in a similar way.

(ii) Hence show that $\frac{\pi}{4}=\arctan\frac{1}{2}+\arctan\frac{1}{3}$ [5]

24. [Maximum mark: 6] ***[without GDC]***

Prove that $\left(\sqrt{3}+\mathrm{i}\right)^n+\left(\sqrt{3}-\mathrm{i}\right)^n$ is real, where $n\in\mathbb{Z}^+$

25. [Maximum mark: 12] ***[without GDC]***

Let $z_1=(1+\mathrm{i}\sqrt{3})^m$ and $z_2=(1-\mathrm{i})^n$.

(a) Find the modulus and argument of z_1 and z_2 in terms of m and n, respectively. [6]

(b) **Hence**, find the smallest positive integers m and n such that $z_1=z_2$. [6]

26. [Maximum mark: 10] ***[without GDC]***

A complex number z is such that $|z|=|z-3\mathrm{i}|$.

(a) Show that the imaginary part of z is $\frac{3}{2}$. [2]

(b) Let z_1 and z_2 be the two possible values of z, such that $|z|=3$.

(i) Sketch a diagram to show the points which represent z_1 and z_2 in the complex plane, where z_1 is in the first quadrant.

(ii) Show that $\arg z_1=\frac{\pi}{6}$.

(iii) Find $\arg z_2$. [4]

(c) Given that $\arg\left(\frac{z_1^k z_2}{2\mathrm{i}}\right)=\pi$, find a value of k. [4]

27. [Maximum mark: 11] ***[without GDC]***

Consider the complex number $z = \dfrac{\left(\cos\frac{\pi}{4} - \mathrm{i}\sin\frac{\pi}{4}\right)^2 \left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)^3}{\left(\cos\frac{\pi}{24} - \mathrm{i}\sin\frac{\pi}{24}\right)^4}$.

(a) (i) Find the modulus of z.

(ii) Find the argument of z, giving your answer in radians. [4]

(b) Using De Moivre's theorem, show that z is a cube root of one, *i.e.* $z^3 = 1$. [2]

(c) Simplify $(1+2z)(2+z^2)$, expressing your answer in the form $a+b\mathrm{i}$, where a and b are **exact** real numbers. [5]

THE EQUATION $z^n = a$

O. Practice questions

28. [Maximum mark: 15] ***[without GDC]***

Complete the following table with the n-th roots of 1

Equation	**Roots**	
	Polar form	**Cartesian form**
$z^2 = 1$	$z_0 = \mathrm{cis}0$ $z_1 = \mathrm{cis}\pi$	$z_0 = 1$ $z_1 = -1$
$z^3 = 1$	$z_0 = \mathrm{cis}0$ $z_1 = \mathrm{cis}\frac{2\pi}{3}$ $z_2 = \mathrm{cis}\frac{4\pi}{3}$	
$z^4 = 1$		
$z^5 = 1$		**(only Polar form here)**
$z^6 = 1$		

29. [Maximum mark: 15] ***[without GDC]***

Solve the following equations and represent the solutions in the complex plane.

(a) $z^3 = -1$, (b) $z^3 = \mathrm{i}$, (c) $z^3 = -\mathrm{i}$

30. [Maximum mark: 16] ***[without GDC]***

Solve the following equations

(a) $z^4 = 16$, (b) $z^4 = -16$, (c) $z^4 = 16\mathrm{i}$ (d) $z^4 = 8\sqrt{2} + 8\sqrt{2}\mathrm{i}$

A. Exam style questions (SHORT)

31. [Maximum mark: 9] ***[without GDC]***

(a) Solve the equation $z^5 = 16\sqrt{2} - 16\sqrt{2}\mathrm{i}$. [6]

(b) Let A, B, C, D, E, be the points in the complex plane that correspond to the five solutions. Find the area of the pentagon ABCDE, in the form $a\sin\theta$. [3]

32. [Maximum mark: 6] ***[without GDC]***

Let $x, y \in \mathbb{R}$, and ω be one of the complex solutions of the equation $z^3 = 1$.

(a) Evaluate: $1 + \omega + \omega^2$ [2]

(b) $(\omega x + \omega^2 y)(\omega^2 x + \omega y)$ in terms of x and y only. [4]

33. [Maximum mark: 6] ***[without GDC]***

Let w be one of the complex solutions of the equation $z^5 = 1$.

(a) Evaluate $1 + w + w^2 + w^3 + w^4$ [2]

(b) Express $1 + 2w + 3w^2 + 4w^3 + 5w^4 + 6w^5 + 7w^6 + 8w^7$ in the form $aw^3 + bw^2 + cw + d$, where a, b, c, d are integers. [4]

34*. [Maximum mark: 12] ***[without GDC]***

Let $w = \cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}$.

(a) Show that w is a root of the equation $z^5 - 1 = 0$. [3]

(b) Show that $(w-1)(w^4 + w^3 + w^2 + w + 1) = w^5 - 1$ and deduce that $w^4 + w^3 + w^2 + w + 1 = 0$. [3]

(c) **Hence**, show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. [6]

35*. [Maximum mark: 10] ***[without GDC]***

(a) Express $z^5 - 1$ as a product of two factors, one of which is linear. [2]

(b) Find the zeros of $z^5 - 1$, giving your answers in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$. [3]

(c) Express $z^4 + z^3 + z^2 + z + 1$ as a product of two real quadratic factors. [5]

36. [Maximum mark: 8] ***[without GDC]***

Given that $z \in \mathbb{C}$,

(a) solve the equation $z^3 - 8\mathrm{i} = 0$, giving your answers in the form $z = r\,\mathrm{cis}\,\theta$.

[**Notice**: in other words, find the cube roots of $8\mathrm{i}$] [6]

(b) show the roots on an Argand diagram. [2]

37. [Maximum mark: 10] ***[without GDC]***

(a) Use de Moivre's theorem to find the roots of the equation $z^4 = 1 - \mathrm{i}$. [6]

(b) Draw these roots on an Argand diagram. [2]

(c) If z_1 is the root in the first quadrant and z_2 is the root in the second quadrant, find $\frac{z_2}{z_1}$ in the form $a + b\mathrm{i}$. [2]

38. [Maximum mark: 6] ***[without GDC]***

The complex number z is defined by

$$z = 4\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right) + 4\sqrt{3}\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)$$

(a) Express z in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where r and θ have exact values.

(b) Find the cube roots of z, expressing in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r\mathrm{e}^{\mathrm{i}\theta}$ have exact values.

B. Exam style questions (LONG)

39. [Maximum mark: 10] ***[without GDC]***

(a) The complex number z is defined by $z = \cos\theta + \mathrm{i}\sin\theta$.

(i) Show that $\frac{1}{z} = \cos(-\theta) + \mathrm{i}\sin(-\theta)$.

(ii) Deduce that $z^n + z^{-n} = 2\cos n\theta$. [5]

(b) (i) Find the binomial expansion of $(z + z^{-1})^5$.

(ii) Hence show that $\cos^5\theta = \frac{1}{16}(a\cos 5\theta + b\cos 3\theta + c\cos\theta)$, where a, b, c are positive integers to be found. [5]

40*. [Maximum mark: 13] ***[without GDC]***

(a) Expand and simplify $(x-1)(x^4+x^3+x^2+x+1)$. [2]

(b) Given that b is a root of the equation $z^5-1=0$ which does not lie on the real axis in the Argand diagram, show that $1+b+b^2+b^3+b^4=0$. [3]

(c) If $u=b+b^4$ and $v=b^2+b^3$ show that

(i) $u+v=uv=-1$;

(ii) $u-v=\sqrt{5}$, given that $u-v>0$. [8]

41. [Maximum mark: 12] ***[without GDC]***

Consider $z^5-32=0$.

(a) Show that $z_1=2\left(\cos\left(\frac{2\pi}{5}\right)+\mathrm{i}\sin\left(\frac{2\pi}{5}\right)\right)$ is one of the complex roots of this equation. [3]

(b) Find $z_1{}^2, z_1{}^3, z_1{}^4, z_1{}^5$, giving your answer in the modulus argument form. [4]

(c) Plot the points that represent $z_1, z_1{}^2, z_1{}^3, z_1{}^4, z_1{}^5$, in the complex plane. [3]

(d) The point $z_1{}^n$ is mapped to $z_1{}^{n+1}$ (where $n = 1, 2, 3, 4$) by a composition of two transformations, Give a full geometric description of the two transformations. [2]

42. [Maximum mark: 9] ***[without GDC]***

(a) Express the complex number $1+\mathrm{i}$ in the form $\sqrt{a}\mathrm{e}^{\mathrm{i}\frac{\pi}{b}}$, where $a,b\in\mathbb{Z}^+$. [2]

(b) Using the result from (a), show that $\left(\frac{1+\mathrm{i}}{\sqrt{2}}\right)^n$, where $n\in\mathbb{Z}$, has only eight distinct values [5]

(c) **Hence** solve the equation $z^n-1=0$. [2]

43*. [Maximum mark: 28] ***[without GDC]***

Let $u=1+\sqrt{3}\mathrm{i}$ and $v=1+\mathrm{i}$, where $\mathrm{i}^2=-1$.

(a) (i) Show that $\frac{u}{v}=\frac{\sqrt{3}+1}{2}+\frac{\sqrt{3}-1}{2}\mathrm{i}$.

(ii) By expressing both u and v in modulus-argument form show that

$$\frac{u}{v}=\sqrt{2}\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right).$$

(iii) **Hence** find the **exact** value of $\tan\frac{\pi}{12}$ in the form $a+b\sqrt{3}$ where $a,b\in\mathbb{Z}$. [15]

(b) Use mathematical induction to prove that for $n\in\mathbb{Z}^+$,

$$\left(1+\sqrt{3}\mathrm{i}\right)^n=2^n\left(\cos\frac{n\pi}{3}+\mathrm{i}\sin\frac{n\pi}{3}\right).$$ [7]

(c) Let $z=\frac{\sqrt{2}v+u}{\sqrt{2}v-u}$.

Show that $\mathrm{Re}\,z=0$. [6]

44*. [Maximum mark: 26] ***[without GDC]***

(a) (i) Factorize $t^3 - 3t^2 - 3t + 1$, giving your answer as a product of a linear factor and a quadratic factor.

(ii) Hence find all the **exact** solutions to the equation $t^3 - 3t^2 - 3t + 1 = 0$. [5]

(b) Using de Moivre's theorem and the binomial expansion.

(i) show that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$;

(ii) write down a similar expression for $\sin 3\theta$. [7]

(c) (i) Hence show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

(ii) Find the values of θ, $0° \le \theta \le 180°$, for which this identity is not valid. [7]

(d) Using th results from parts (a) and (c), find the **exact** values of $\tan 15°$ and $\tan 75°$. [7]

45*. [Maximum mark: 20] ***[without GDC]*** ***[needs differentiation]***

Let $y = \cos\theta + \mathrm{i}\sin\theta$

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = \mathrm{i}y$ *[*i *may be treated in the same way as a real constant]* [3]

(b) Solve the differential equation in (a) to show that $y = \mathrm{e}^{\mathrm{i}\theta}$. [5]

(c) Use this result to deduce de Moivre's theorem. [2]

(d) (i) Given that $\dfrac{\sin 6\theta}{\sin\theta} = a\cos^5\theta + b\cos^3\theta + c\cos\theta$, where $\sin\theta \ne 0$, use de Moivre's theorem with $n = 6$ to find the values of the constants a, b and c.

(ii) Hence deduce the value of $\lim\limits_{\theta\to 0}\dfrac{\sin 6\theta}{\sin\theta}$. [10]

46. [Maximum mark: 13] ***[without GDC]*** ***[needs integration]***

Consider the complex number $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Using De Moivre's theorem show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$. [2]

(b) By expanding $\left(z + \dfrac{1}{z}\right)^4$ show that $\cos^4\theta$ = $\dfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ [4]

(c) Find $\int_0^a \cos^4\theta \mathrm{d}\theta$. [4]

(d) Find $\cos^4 15°$. [3]

INDUCTION

Please look at also the **COMPLEX NUMBERS** section from the **EXERCISES**

[Math-AA 1.9] MATHEMATICAL INDUCTION