INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

Math AA

EXERCISES [Math-AA 1.11-1.12]

COMPLEX NUMBERS (CARTESIAN FORM)

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CARTESIAN FORM

O. Practice questions

1. [Maximum mark: 13] ***[without GDC]***

Complete the following table

z	$\mathrm{Re}(z)$	$\mathrm{Im}(z)$	z^*	$\lvert z\rvert$	$z \cdot z^*$
$2+3\mathrm{i}$	2	3	$2-3\mathrm{i}$	$\sqrt{2^2+3^2}=\sqrt{13}$	$(2+3\mathrm{i})(2-3\mathrm{i})=4+9=13$
$3+2\mathrm{i}$					
$\mathrm{i}-1$					
$3+4\mathrm{i}$					
$3-4\mathrm{i}$					
$-3+4\mathrm{i}$					
$-3-4\mathrm{i}$					
$2\mathrm{i}$					
$-2\mathrm{i}$					
2					
-2					
0					
$a+b\mathrm{i}$					
$a-b\mathrm{i}$					

2. [Maximum mark: 12] ***[without GDC]***

Let $z_1 = 10 + 5\mathrm{i}$ and $z_2 = 3 + 4\mathrm{i}$

(a) Find the following results in the form $x + y\mathrm{i}$.

$z_1 + z_2$	
$z_1 - z_2$	
$z_1 z_2$	
$\frac{z_1}{z_2}$	

[6]

(b) Find the following values

$\lvert z_1 \rvert$		$\lvert z_2 \rvert$	
$\lvert z_1 + z_2 \rvert$		$\lvert z_1 - z_2 \rvert$	
$\frac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$		$\left\lvert \frac{z_1}{z_2} \right\rvert$	

[6]

[Confirm the results with your GDC]

3. [Maximum mark: 6] ***[without GDC]***

Find the following values

i^2		i^3	
i^4		i^5	
i^8		i^{10}	
i^{100}		i^{101}	
i^{53}		i^{111}	
$(2\mathrm{i})^3$		$(2\mathrm{i})^4$	

[Confirm the results with your GDC]

4. [Maximum mark: 6] ***[without GDC]***

Let $z = 2 + 3\mathrm{i}$. Express the following powers of z in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{Z}$

(i) z^2 (ii) z^3 (iii) z^4

[Confirm the results with your GDC]

5. [Maximum mark: 6] ***[without GDC]***

(a) Find $(1 - \mathrm{i}\sqrt{3})^2$ in the form $x + y\mathrm{i}$, where $x, y \in \mathbb{R}$. [3]

(b) Find $(1 - \mathrm{i}\sqrt{3})^3$ in the form $x + y\mathrm{i}$, where $x, y \in \mathbb{R}$. [3]

[Confirm the results with your GDC]

6. [Maximum mark: 6] ***[without GDC]***

Let $f(z) = z^2 - 8z + 20$.

(a) Find the discriminant Δ of the quadratic function f. [1]

(b) Find the complex roots of the equation $f(z) = 0$ in the form $z = x \pm y\mathrm{i}$ [3]

(c) Use factorisation to express f in the form $f(z) = (z - h)^2 + k$. [2]

[Confirm question (b) with your GDC]

7. [Maximum mark: 4] ***[with / without GDC]***

Find the complex roots of the quadratic equations

(i) $z^2 + 4 = 0$. (ii) $z^2 + 3 = 0$

8. [Maximum mark: 10] ***[with / without GDC]***

Let $f(z) = 4z^2 - 8z + 13$.

(a) Find the discriminant Δ of the quadratic function f. [1]

(b) Find the complex roots of the equation $f(z) = 0$, in the form $z = x \pm y\mathrm{i}$. [3]

(c) Confirm that the sum S and the product P of the roots are given by

(i) $S = -\dfrac{b}{a}$. (ii) $P = \dfrac{c}{a}$. [4]

(d) By using factorisation, express f in the form $f(z) = a(z - h)^2 + k$. [2]

[Confirm question (b) with your GDC]

9. [Maximum mark: 4]

(a) Given that $(a - 2) + 3\mathrm{i} = 7 + (b - 1)\mathrm{i}$, find the value of a and of b, where $a, b \in \mathbb{Z}$. [2]

(b) Given that $(c - 2) + (d - 1)\mathrm{i} = 0$, find the value of c and of d, where $c, d \in \mathbb{Z}$. [2]

10. [Maximum mark: 9] ***[without GDC]***

Let $z = x + y\mathrm{i}$. Find the values of x and y if $(2+5\mathrm{i})\,z = 1+17\mathrm{i}$

(a) by using division (an equation of the form $az = b$, implies $z = b/a$) [4]

(b) by substituting $z = x + y\mathrm{i}$ in the equation and solving the simultaneous equations. [5]

11. [Maximum mark: 10] ***[without GDC]***

Solve the equations

(a) $(2+5\mathrm{i})z + 9 = 3z + 19\mathrm{i}$ [5]

(b) $(2+5\mathrm{i})z + 8 = 3\bar{z} + 20\mathrm{i}$ [5]

A. Exam style questions (SHORT)

12. [Maximum mark: 6] ***[without GDC]***

Let the complex number z be given by $z = 1 + \dfrac{\mathrm{i}}{\mathrm{i} - \sqrt{3}}$.

Express z in the form $a + b\mathrm{i}$, giving the **exact** values of the real constants a, b.

[Confirm question (b) with your GDC]

13. [Maximum mark: 5] ***[without GDC]***

Express $\dfrac{1}{(1-i\sqrt{3})^3}$ in the form $\dfrac{a}{b}$ where $a, b \in Z$.

[Confirm question (b) with your GDC]

14. [Maximum mark: 6] ***[without GDC]***

Let $z = \dfrac{2}{1-\mathrm{i}} + 1 - 4\mathrm{i}$. Express z^2 in the form $x + y\mathrm{i}$ where $x, y \in Z$.

[Confirm question (b) with your GDC]

15. [Maximum mark: 6] ***[without GDC]***

Consider the equation $2(p + \mathrm{i}q) = q - \mathrm{i}p - 2(1-\mathrm{i})$, where p and q are both real numbers. Find p and q.

16. [Maximum mark: 6] ***[without GDC]***

Given that $(a+\mathrm{i})\,(2-b\mathrm{i}) = (7-\mathrm{i})$, find the value of a and of b, where $a, b \in \mathbb{Z}$.

17. [Maximum mark: 4] ***[with / without GDC]***

Find the values of a and b, where a and b are real, given that $(a+b\mathrm{i})(2-\mathrm{i}) = 5-\mathrm{i}$

18. [Maximum mark: 4] ***[with / without GDC]***

Let $z = x + y\mathrm{i}$. Find the values of x and y if $(1-\mathrm{i})z = 1 - 3\mathrm{i}$.

19. [Maximum mark: 5] ***[without GDC]***

The complex number z satisfies $\mathrm{i}(z+2) = 1 - 2z$, where $\mathrm{i} = \sqrt{-1}$. Write z in the form $z = a + b\mathrm{i}$, where a and b are real numbers.

20. [Maximum mark: 6] ***[without GDC]***

The two complex numbers $z_1 = \frac{a}{1+i}$ and $z_2 = \frac{b}{1-2i}$ where $a, b \in R$, are such that $z_1 + z_2 = 3$. Calculate the value of a and of b.

21. [Maximum mark: 6] ***[without GDC]***

Solve the following equation for z, where z is a complex number.

$$\frac{z}{3+4\mathrm{i}} + \frac{z-1}{5\mathrm{i}} = \frac{5}{3-4\mathrm{i}}$$

Give your answer in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{Z}$

22. [Maximum mark: 6] ***[without GDC]***

Given that $|z| = 2\sqrt{5}$, find the complex number z that satisfies the equation

$$\frac{25}{z} - \frac{15}{z^*} = 1 - 8\mathrm{i}.$$

23. [Maximum mark: 6] ***[with / without GDC]***

Let z_1 and z_2 be complex numbers. Solve the simultaneous equations

$$2z_1 + 3z_2 = 7$$
$$z_1 + \mathrm{i}z_2 = 4 + 4\mathrm{i}$$

Give your answers in the form $z = a + b\mathrm{i}$, where $a, b \in \mathbb{Z}$.

24*. [Maximum mark: 7] ***[without GDC]***

Consider $w = \frac{z}{z^2 + 1}$ where $z = x + y\mathrm{i}$, $y \neq 0$ and $z^2 + 1 \neq 0$.

Given that $\mathrm{Im}\, w = 0$, show that $|z| = 1$.

25*. [Maximum mark: 6] ***[without GDC]***

If $z = x + y\mathrm{i}$ is a complex number and $|z + 16| = 4|z + 1|$, find the value of $|z|$.

POLYNOMIALS

O. Practice questions

26. [Maximum mark: 10] ***[with GDC]***

For a cubic polynomial $f(z)=a_3z^3+a_2z^2+a_1x+a_0$, we know that the **sum** S and the **product** P of the roots are given by $S=-\frac{a_2}{a_3}$ and $P=-\frac{a_0}{a_3}$.

Let $f(z)=z^3-3z^2+7z-5$

(a) Find the **real root** and the two **complex roots** of $f(z)$ (using GDC). [2]

(b) Confirm the formulas above for the sum S and the product P of the roots. [4]

(c) Write down the three linear factors of $f(z)$ (with complex coefficients). [2]

(d) **Hence**, express $f(z)$ in the form $(z-a)(z^2+bz+c)$ where $a,b,c\in Z$. [2]

27. [Maximum mark: 10] ***[with GDC]***

For the polynomial $f(z)=a_4z^4+a_3z^3+a_2z^2+a_1x+a_0$, we know that the **sum** S and the **product** P of the roots are given by $S=-\frac{a_3}{a_4}$ and $P=\frac{a_0}{a_4}$.

Let $f(z)=2z^4-10z^3+26z^2-38z+20$

(a) Find the two **reals roots** and the two **complex roots** of $f(z)$ (using GDC). [2]

(b) Confirm the formulas above for the sum S and the product P of the roots. [4]

(c) Write down the four linear factors of $f(z)$ (with complex coefficients). [2]

(d) Express $f(z)$ in the form $a(z-p)(x-q)(z^2+bz+c)$ where $a,p,q,b,c\in Z$ [2]

28. [Maximum mark: 8] ***[with / without GDC]***

Consider the polynomial $f(z)=2z^4+az^3+26z^2+bz+20$.

Given that 1 and 2 are roots of the polynomial

(a) find the values of a and b. [4]

(b) find the other two roots of $f(z)$. [4]

29. [Maximum mark: 6] ***[without GDC]***

Consider the polynomial $f(z)=2z^4-10z^3+26z^2-38z+20$

Given that $z=1-2\mathrm{i}$ is a root find the other 3 roots of $f(z)$.

30. [Maximum mark: 6] ***[without GDC]***

Find the polynomial $f(z)$ of degree 4, given that

- 3 of the zeros of the polynomial are $1-2\mathrm{i}$, 1, 2
- $f(z)$ leave a remainder 96 when it is divided by $z+1$.

A. Exam style questions (SHORT)

31. [Maximum mark: 5] ***[without GDC]***

$(z+2\text{i})$ is a factor of $2z^3-3z^2+8z-12$. Find the other two factors.

32. [Maximum mark: 6] ***[without GDC]***

Let $P(z)=z^3+az^2+bz+c$, where a, b and $c \in \mathbb{R}$. Two of the roots of $P(z)=0$ are -2 and $(-3+2\text{i})$. Find the value of a, of b and of c.

33. [Maximum mark: 6] ***[without GDC]***

The polynomial $P(z)=z^3+mz^2+nz-8$ is divisible by $(z+1+\text{i})$, where $z \in C$ and $m, n \in R$. Find the value of m and of n.

METHOD A: Use $P(-1-\text{i})=0$ [not the ideal way; just for practice!]

METHOD B: Find first the three roots of $P(z)$, and use factorization [Much quicker!]

B. Exam style questions (LONG)

34. [Maximum mark: 10] ***[without GDC]***

(a) Evaluate $(1+\text{i})^2$, where i = $\sqrt{-1}$. [2]

(b) Prove, by mathematical induction, that $(1+\text{i})^{4n}=(-4)^n$, where $n \in \mathbb{N}^*$. [6]

(c) Hence or otherwise, find $(1+\text{i})^{32}$. [2]

35. [Maximum mark: 13] ***[without GDC]***

Let $z=a+b\text{i}$ and $w=c+d\text{i}$,

(a) Express zw in the form $x+y\text{i}$ [2]

(b) Show that $|zw|^2=(ac)^2+(bd)^2+(ac)^2+(bd)^2$ [2]

(c) Show that $\overline{z+w}=\overline{z}+\overline{w}$ [2]

(d) Show that $\overline{zw}=\overline{z}\,\overline{w}$ [3]

(e) Show that $|zw|=|z||w|$ [4]

36. [Maximum mark: 12] ***[without GDC]***

It is given that $\overline{zw}=\overline{z}\,\overline{w}$ and $|zw|=|z||w|$ for any complex numbers z and w.
Show, by using mathematical induction, that for any $n \geq 2$ it holds

(a) $\overline{z^n}=\overline{z}^{\,n}$ [6]

(b) $|z^n|=|z|^n$ [6]