INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

Math AA

EXERCISES [Math-AA 2.13-2.14]

RATIONAL FUNCTIONS – INEQUALITIES

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O. Practice questions

1. [Maximum mark: 18] ***[without GDC]***

For the following functions write down the roots, the y-intercept, the vertical and horizontal asymptotes (if there exist), the domain and the range.

	$f(x)=\frac{3x+8}{2x+7}$	$f(x)=\frac{8}{2x+7}$	$f(x)=\frac{3x+8}{7}$
Roots			
y **-intercept**			
V.A.			
H.A.			
Domain			
Range			

2. [Maximum mark: 15] ***[without GDC]***

For the following functions write down the roots, the y-intercept, the vertical and horizontal asymptotes and the domain.

	$f(x)=\frac{(x-3)(x-4)}{(x+1)(x-2)}$	$f(x)=\frac{(2x-3)(x-4)}{(x+1)(x-2)}$	$f(x)=\frac{2x-3}{(x+1)(x-2)}$
Roots			
y **-intercept**			
V.A.			
H.A.			
Domain			

3. [Maximum mark: 10] ***[without GDC]***

For the following functions, find all the asymptotes (horizontal, vertical, or oblique).

(a) $f(x)=\dfrac{4x^2+4x+1}{2x+3}$ [4]

(b) $f(x)=\dfrac{2x^3+5x^2+4x+1}{x^2-x-2}$ [6]

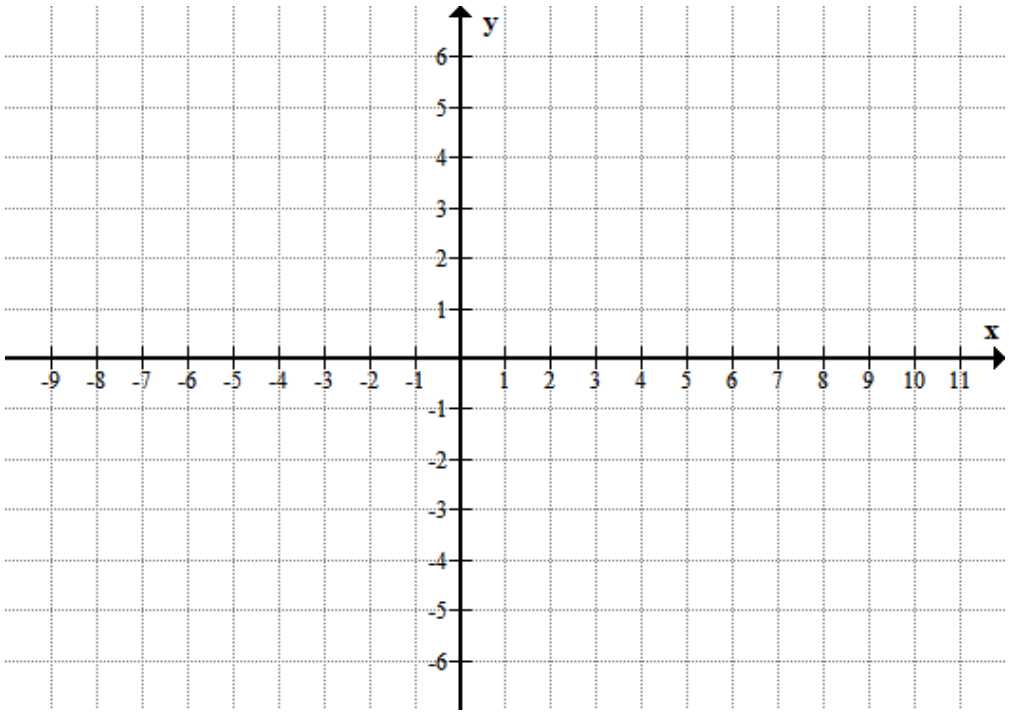
4. [Maximum mark: 10] ***[without GDC]***

Let $f(x)=\dfrac{(x-1)(x-6)}{(x+2)(x-3)}$.

(a) Solve the equation $f(x)=1$. [3]

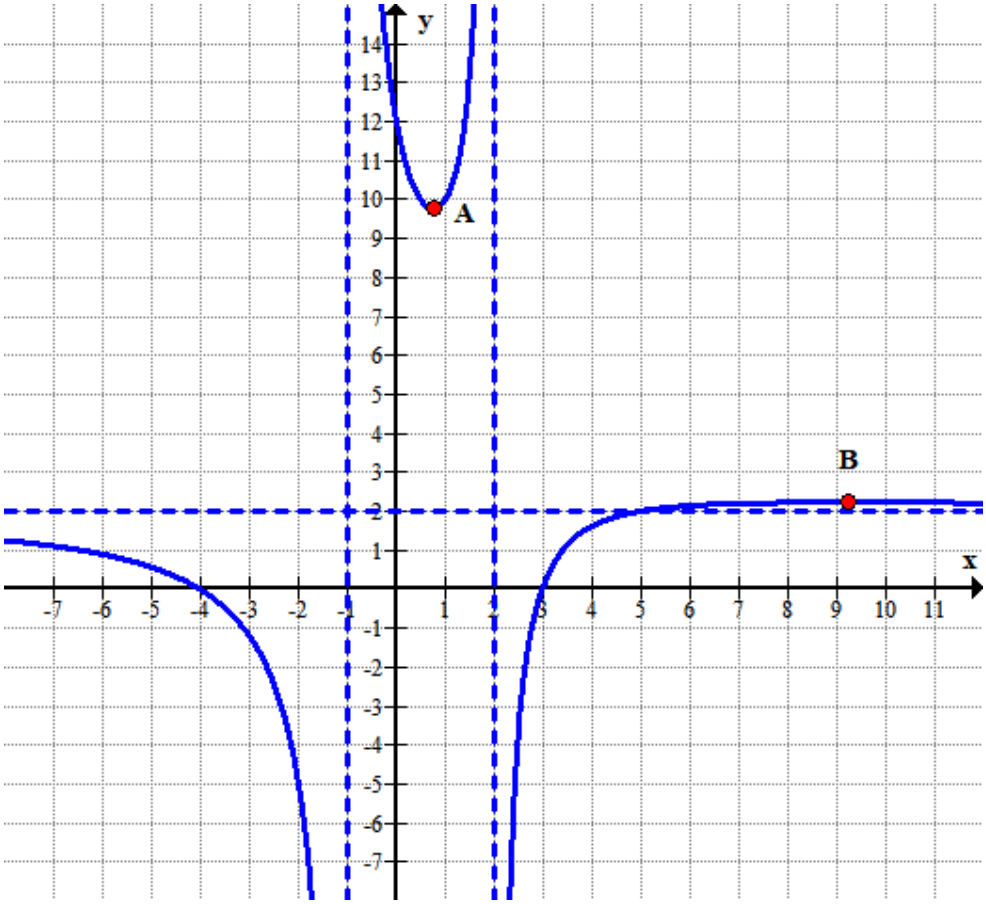
(b) On the diagram below, sketch the graph of $y=f(x)$. Indicate all the asymptotes, the $x-$ and the $y-$intercepts, and the information found in (a). [5]

(c) Hence, write down the domain and the range of f. [2]



5. [Maximum mark: 7] ***[with GDC]***

The following diagram shows part of the graph of $f(x)=\dfrac{2x^2+2x-24}{x^2-x-2}$



(a) Express $f(x)$ in the form $f(x)=\dfrac{2(x+a)(x-b)}{(x+c)(x-d)}$, where $a, b, c, d \in \mathbb{Z}^+$. [2]

(b) Write down the equations of the horizontal and the vertical asymptotes. [2]

There is a local minimum at point A and a local maximum at point B as shown above.

(c) Write down the coordinates of A and B and hence the range of $f(x)$. [3]

6. [Maximum mark: 12] ***[without GDC]***

Let $f(x)=\dfrac{(x+2)(x-3)}{3x-6}$

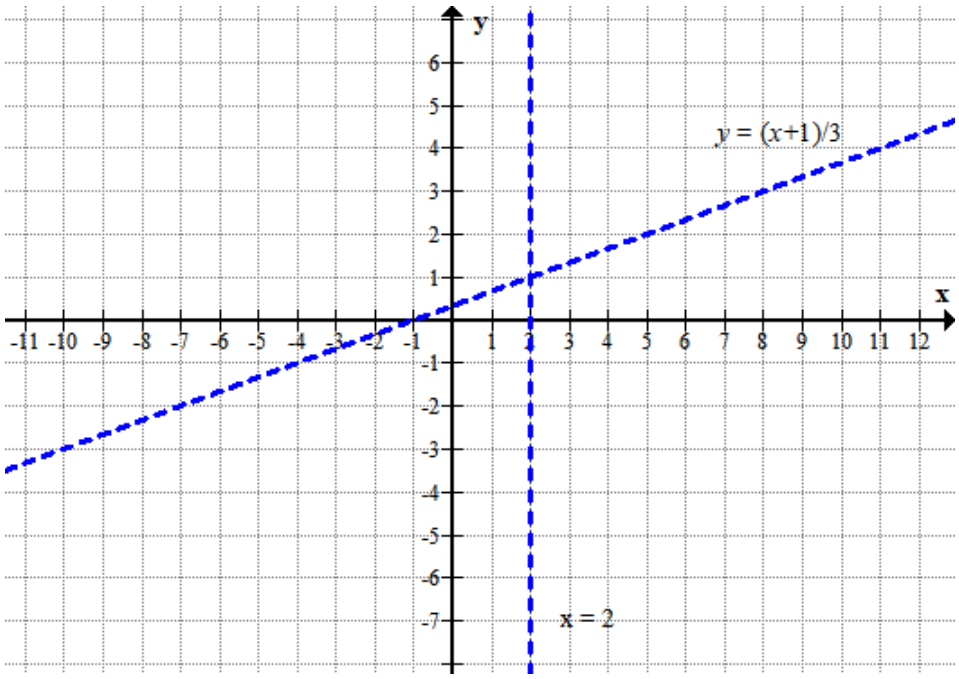
(a) Show that graph of $y=f(x)$ has an oblique asymptote at the line $y=\dfrac{x+1}{3}$ [3]

(b) Solve the equation $f(x)=1$. [3]

(c) On the diagram below, sketch the graph of $y=f(x)$. The asymptotes are shown.

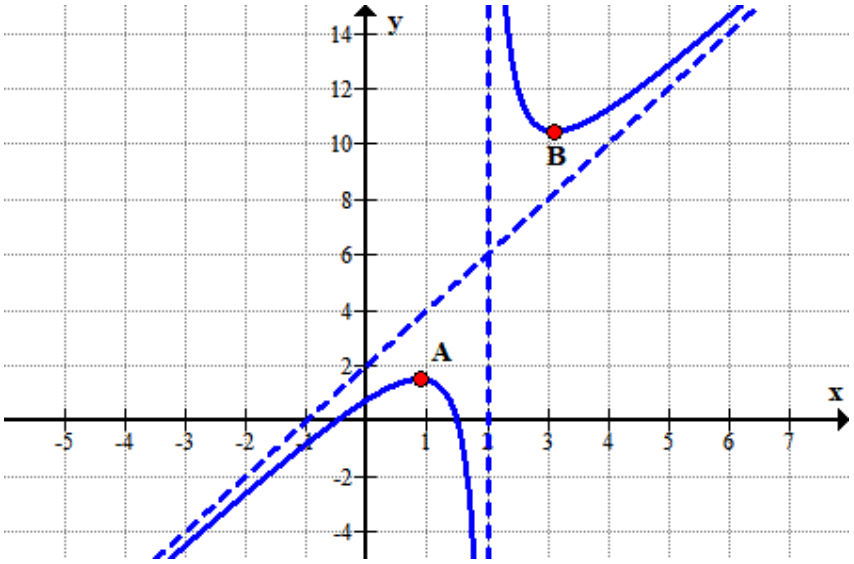
Indicate the $x-$ and the $y-$intercepts and the information found in (b). [4]

(d) Hence, write down the domain and the range of f. [2]



7. [Maximum mark: 10] ***[with GDC]***

The following diagram shows part of the graph of $f(x)=\dfrac{4x^2-4x-3}{2x-4}$



(a) Express $f(x)$ in the form $f(x)=\dfrac{(2x+a)(2x-b)}{2x-4}$, where $a, b \in \mathbb{Z}^+$. [2]

(b) Write down the x-intercepts and the equation of the vertical asymptote. [2]

(c) Find the equation of the oblique asymptote. [3]

There is a local maximum at point A and a local minimum at point B as shown above.

(d) Write down the coordinates of A and B and **hence** the range of $f(x)$. [3]

8. [Maximum mark: 12] ***[with / without GDC]***

Express the following rational functions in partial fractions.

(a) $f(x)=\dfrac{6}{x^2-x-2}$ (b) $f(x)=\dfrac{3x+6}{x^2-x-2}$ (c) $f(x)=\dfrac{x}{2x^2-2x-4}$

9. [Maximum mark: 10] ***[without GDC]***

Consider the polynomial

$$f(x)=(x-1)(x-2)(x-3)(x-4)(x-5)$$

(a) Complete the sign table below (Indicate + or – in each interval)

x	-∞	1	2	3	4	5	+∞
$f(x)$							

[2]

(b) Solve the inequality $f(x)>0$. [2]

(c) Solve the inequality $f(x)\geq 0$. [2]

(d) Solve the inequality $\dfrac{(x-1)(x-3)(x-5)}{(x-2)(x-4)}>0$. [2]

(e) Solve the inequality $\dfrac{(x-1)(x-3)(x-5)}{(x-2)(x-4)}\geq 0$. [2]

10. [Maximum mark: 13] ***[without GDC]***

Consider the following functions.

$$g(x)=(x-1)(x-2)(x-3)(x-4)^2(x-5)$$

(a) Complete the sign table below (Indicate + or – in each interval)

x	-∞	1	2	3	4	5	+∞
$g(x)$							

[2]

(b) Solve the inequality $g(x)>0$. [2]

(c) Solve the inequality $g(x)\geq 0$. [2]

(d) Solve the inequality $g(x)<0$. [2]

(e) Solve the inequality $g(x)\leq 0$. [2]

(f) Solve the inequality $\dfrac{(x-1)(x-3)(x-5)}{(x-2)(x-4)^2}\geq 0$. [3]

11. [Maximum mark: 8] ***[without GDC]***

Solve the inequalities (notice that they all involve the same factors).

(i) $\dfrac{5(x-1)(x-2)^2}{(x-3)^3}\geq 0$ (ii) $\dfrac{(x-1)(x-3)^3}{5(x-2)^2}\geq 0$ (iii) $\dfrac{(x-2)^2(x-3)^3}{5(x-1)}\geq 0$

12. [Maximum mark: 6] ***[with / without GDC]***

Solve the inequality $x+\frac{2}{x}\geq 3$

13. [Maximum mark: 9] ***[with / without GDC]***

(a) Solve the equation $\frac{2x-5}{x+3}=1$. [2]

(b) Solve the inequality $\frac{2x-5}{x+3}\geq 1$. [4]

(c) Hence, solve (i) $\frac{2x-5}{x+3}>1$ (i) $\frac{2x-5}{x+3}\leq 1$ (i) $\frac{2x-5}{x+3}<1$ [3]

14. [Maximum mark: 10] ***[with / without GDC]***

(a) Solve the equation $\frac{2x+5}{x+13}=\frac{x}{x+1}$. [4]

(b) Solve the inequality $\frac{2x+5}{x+13}\leq\frac{x}{x+1}$. [6]

15*. [Maximum mark: 14] ***[without GDC]***

The polynomial $f(x)=x^3-7x^2+ax-9$ is divisible by $(x-1)$.

(a) Find the value of a. [3]

(b) Give full factorization of $f(x)$. [5]

(c) Solve the inequalities

(i) $f(x)>0$ (ii) $f(x)<0$ (iii) $f(x)\geq 0$ (iv) $f(x)\leq 0$ [6]

A. Exam style questions (SHORT)

16. [Maximum mark: 5] ***[without GDC]***

Find all the asymptotes (horizontal, vertical, or oblique) of $f(x)=\frac{3x^2-x+1}{2x^2-14x+24}$

17. [Maximum mark: 5] ***[without GDC]***

Find all the asymptotes (horizontal, vertical, or oblique) of $f(x)=\frac{3x^2-x+1}{x^2-x+1}$

18. [Maximum mark: 6] ***[without GDC]***

Find all the asymptotes (horizontal, vertical, or oblique) of $f(x)=\frac{6x^2+5x+1}{3x+7}$

19. [Maximum mark: 7] ***[without GDC]***

Find all the asymptotes (horizontal, vertical, or oblique) of $f(x)=\dfrac{6x^3+1}{2x^2-14x+24}$

20. [Maximum mark: 7] ***[without GDC]***

Express in partial fractions the function $f(x)=\dfrac{5}{2x^2-14x+24}$

21. [Maximum mark: 7] ***[without GDC]***

Express in partial fractions the function $f(x)=\dfrac{5x+1}{2x^2-14x+24}$

22. [Maximum mark: 10] ***[without GDC]***

(a) Show that $x^3-x^2-x+1=(x-1)^2(x+1)$ [2]

(b) Show that $x^4-3x^3+x^2+3x-2=(x-1)^2(x+1)(x-2)$ [2]

(c) **Hence**, solve the inequalities

(i) $x^4-3x^3+x^2+3x-2\geq 0$

(ii) $\dfrac{x^3-x^2-x+1}{x-2}>0$ (iii) $\dfrac{x^3-x^2-x+1}{x-2}\geq 0$ [6]

23. [Maximum mark: 8] ***[without GDC]***

(a) Show that $4x^3-3x+1=(2x-1)^2(x+1)$ [2]

(b) **Hence**, solve the inequalities

(i) $\dfrac{4x^3-3x+1}{2-x}\geq 0$ (ii) $\dfrac{4x^3-3x+1}{2-x}\leq 0$. [6]

Notice

Question **23**(a) could also be given as follows:

The cubic polynomial $f(x)=4x^3-3x+1$ is divisible be $(2x-1)^2$. Find the 3 roots of f and hence factorize $f(x)$ completely.

24. [Maximum mark: 8] ***[with / without GDC]***

(a) Solve the equation $\dfrac{2x^2-x-36}{x-2}=3$. [4]

(b) Solve the inequality $\dfrac{2x^2-x-36}{x-2}\geq 3$. [4]

25. [Maximum mark: 8] ***[with / without GDC]***

Let $f(x)=\frac{x+4}{x+1}$, $x\neq -1$ and $g(x)=\frac{x-2}{x-4}$, $x\neq 4$.

(a) Solve the equation $f(x)=g(x)$. [3]

(b) Find the set of values of x such that $f(x)\leq g(x)$. [5]

26. [Maximum mark: 5] ***[with GDC]***

Solve the inequality $x^2-4+\frac{3}{x}<0$.

27*. [Maximum mark: 8] ***[without GDC]***

Solve the inequality $x^2-4+\frac{3}{x}<0$ given that one of the zeros of the function is $x=1$.

28. [Maximum mark: 9] ***[with / without GDC]***

Let $f(x)=\sqrt{\frac{8x-4}{x-3}}$.

(a) Find the largest set of values of x such that the function f takes real values. [5]

(b) Find $f\left(\frac{1}{2}\right)$ and hence write down the range of f. [2]

(c) Write down

(i) the equation of the vertical asymptote;

(ii) the equation of the horizontal asymptote. [2]

29. [Maximum mark: 7] ***[with / without GDC]***

The functions f and g are defined by

$$f(x)=2x-1, \qquad g(x)=\frac{x}{x+1},\ x\neq -1$$

Find the values of x for which $(f\circ g)(x)\leq (g\circ f)(x)$.

30. [Maximum mark: 8] ***[without GDC]***

The polynomial $f(x)=x^3-4x^2+3x+a$ is divisible by $(x-1)$.

(a) Find the value of a. [3]

(b) Give full factorization of $f(x)$. [3]

(c) Solve the inequality $f(x)\leq 0$. [2]

31. [Maximum mark: 8] ***[without GDC]***

The polynomial $f(x) = x^3 - 2x^2 + x + a$ is divisible by $(x-1)$.

(a) Find the value of a. [3]

(b) Give full factorization of $f(x)$. [3]

(c) Solve the inequality $f(x) \le 0$. [2]

32. [Maximum mark: 8] ***[without GDC]***

When $f(x) = x^3 + x^2 + x + a$ is divided by $(x-1)$ the remainder is 3.

(a) Find the value of a. [3]

(b) Give full factorization of $f(x)$. [3]

(c) Solve the inequality $f(x) \le 0$. [2]

33. [Maximum mark: 6] ***[without GDC]***

The polynomial $f(x) = 3x^3 - a$ is divisible by $(x-1)$.

(a) Find the value of a. [3]

(b) Solve the inequality $f(x) \le 0$. [3]

B. Exam style questions (LONG)

34. [Maximum mark: 24] ***[with / without GDC]***

Let $f(x) = \dfrac{6}{x^2 - 5x + 4}$, $g(x) = \dfrac{x^2 - 5x + 10}{x^2 - 5x + 4}$, $h(x) = \dfrac{x^3 - 5x^2 + 4x + 12}{x^2 - 5x + 4}$

(a) Find the equations of all the asymptotes for the graphs of

(i) $y = f(x)$ (ii) $y = g(x)$ (iii) $y = h(x)$ [8]

(b) Express $f(x)$ in partial fractions [5]

(c) Express $g(x)$ in the form $g(x) = q + \dfrac{p}{x^2 - 5x + 4}$, where $p, q \in \mathbb{Z}$ [2]

(d) **Hence** express

(i) $g(x)$ in the form $g(x) = C + \dfrac{A}{(x-a)} + \dfrac{B}{(x-b)}$

(ii) $h(x)$ in the form $h(x) = Ex + \dfrac{C}{(x-c)} + \dfrac{D}{(x-d)}$ [4]

(e) Solve the inequality $5f(x) \le 3g(x)$ [5]

35. [Maximum mark: 13] ***[with / without GDC]***

Let $f(x)=\dfrac{x^3+50}{x^2-6x+8}, \quad x\neq 2, x\neq 4$

(a) Find the quotient and the remainder of the long division of x^3+50 by x^2-6x+8 [3]

(b) **Hence,** express $f(x)$ in the form $f(x)=q(x)+\dfrac{ax+b}{x^2-6x+8}$ [2]

(c) Write down the equations of all the asymptotes of $y=f(x)$ [3]

(d) Express $f(x)$ in the form $f(x)=q(x)+\dfrac{A}{x-2}+\dfrac{B}{x-4}$ [5]