Test 3 [43 marks]

1. [Maximum mark: 5] EXN.1.SL.TZ0.2

Solve the equation $2\ln x = \ln 9 + 4$. Give your answer in the form $x = pe^q$ where $p, q \in \mathbb{Z}^+$. [5]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

METHOD 1

$2\ln x - \ln 9 = 4$

uses $m\ln x = \ln x^m$ **(M1)**

$\ln x^2 - \ln 9 = 4$

uses $\ln a - \ln b = \ln\frac{a}{b}$ **(M1)**

$\ln\frac{x^2}{9} = 4$

$\frac{x^2}{9} = e^4$ **A1**

$x^2 = 9e^4 \Rightarrow x = \sqrt{9e^4}\ (x > 0)$ **A1**

$x = 3e^2\ (p = 3,\ q = 2)$ **A1**

METHOD 2

expresses 4 as $4\ln e$ and uses $\ln x^m = m\ln x$ **(M1)**

$2 \ln x = 2 \ln 3 + 4 \ln \mathrm{e} \;\; (\ln x = \ln 3 + 2 \ln \mathrm{e})$ **A1**

uses $2 \ln \mathrm{e} = \ln \mathrm{e}^2$ and $\ln a + \ln b = \ln ab$ **(M1)**

$\ln x = \ln \left(3\mathrm{e}^2\right)$ **A1**

$x = 3\mathrm{e}^2 \;\; (p = 3,\; q = 2)$ **A1**

METHOD 3

expresses 4 as $4 \ln \mathrm{e}$ and uses $m \ln x = \ln x^m$ **(M1)**

$\ln x^2 = \ln 3^2 + \ln \mathrm{e}^4$ **A1**

uses $\ln a + \ln b = \ln ab$ **(M1)**

$\ln x^2 = \ln \left(3^2 \mathrm{e}^4\right)$

$x^2 = 3^2 \mathrm{e}^4 \Rightarrow x = \sqrt{3^2 \mathrm{e}^4} \;\; (x > 0)$ **A1**

so $x = 3\mathrm{e}^2 \;\; (x > 0) \;\; (p = 3,\; q = 2)$ **A1**

[5 marks]

2. [Maximum mark: 6] 22M.1.SL.TZ2.3

Consider any three consecutive integers, $n - 1$, n and $n + 1$.

(a) Prove that the sum of these three integers is always divisible by 3. [2]

Markscheme

$(n - 1) + n + (n + 1)$ ***(A1)***

$= 3n$ ***A1***

which is always divisible by 3 ***AG***

[2 marks]

(b) Prove that the sum of the squares of these three integers is never divisible by 3. [4]

Markscheme

$(n-1)^2 + n^2 + (n+1)^2 \quad (= n^2 - 2n + 1 + n^2 + n^2 + 2n + 1)$

A1

attempts to expand either $(n-1)^2$ or $(n+1)^2$ (do not accept $n^2 - 1$ or $n^2 + 1$) ***(M1)***

$= 3n^2 + 2$ ***A1***

demonstrating recognition that 2 is not divisible by 3 or $\frac{2}{3}$ seen after correct expression divided by 3 ***R1***

$3n^2$ is divisible by 3 and so $3n^2 + 2$ is never divisible by 3

OR the first term is divisible by 3, the second is not

OR $3\left(n^2 + \frac{2}{3}\right)$ OR $\frac{3n^2+2}{3} = n^2 + \frac{2}{3}$

hence the sum of the squares is never divisible by 3 ***AG***

[4 marks]

3. [Maximum mark: 6] 22M.1.AHL.TZ1.9

Consider the complex numbers $z_1 = 1 + b\mathrm{i}$ and $z_2 = \left(1 - b^2\right) - 2b\mathrm{i}$, where $b \in \mathbb{R},\ b \neq 0$.

(a) Find an expression for $z_1 z_2$ in terms of b. [3]

Markscheme

$z_1 z_2 = (1 + b\mathrm{i})\left(\left(1 - b^2\right) - (2b)\mathrm{i}\right)$

$= \left(1 - b^2 - 2\mathrm{i}^2 b^2\right) + \mathrm{i}\left(-2b + b - b^3\right)$ ***M1***

$= \left(1 + b^2\right) + \mathrm{i}\left(-b - b^3\right)$ ***A1A1***

Note: Award ***A1*** for $1 + b^2$ and A1 for $-b\mathrm{i} - b^3\mathrm{i}$.

[3 marks]

(b) Hence, given that $\arg(z_1 z_2) = \frac{\pi}{4}$, find the value of b. [3]

Markscheme

$\arg(z_1 z_2) = \arctan\left(\frac{-b-b^3}{1+b^2}\right) = \frac{\pi}{4}$ ***(M1)***

EITHER

$\arctan(-b) = \frac{\pi}{4}$ (since $1 + b^2 \neq 0$, for $b \in \mathbb{R}$) ***A1***

OR

$-b - b^3 = 1 + b^2$ (or equivalent) ***A1***

THEN

$b = -1$ ***A1***

[3 marks]

4. [Maximum mark: 5] 22M.1.AHL.TZ2.9

Prove by contradiction that the equation $2x^3 + 6x + 1 = 0$ has no integer roots. [5]

Markscheme

METHOD 1 (rearranging the equation)

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$ ***M1***

Note: Award ***M1*** for equivalent statements such as 'assume that α is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of x throughout the proof.

Award ***M1*** for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$.

Note: Award ***M0*** for statements such as "let's consider the equation has integer roots…", "let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$…"

Note: Subsequent marks after this ***M1*** are independent of this ***M1*** and can be awarded.

attempts to rearrange their equation into a suitable form ***M1***

EITHER

$2\alpha^3 + 6\alpha = -1$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow 2\alpha^3 + 6\alpha$ is even ***R1***

$2\alpha^3 + 6\alpha = -1$ which is not even and so α cannot be an integer ***R1***

Note: Accept '$2\alpha^3 + 6\alpha = -1$ which gives a contradiction'.

OR

$1 = 2\left(-\alpha^3 - 3\alpha\right)$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow \left(-\alpha^3 - 3\alpha\right) \in \mathbb{Z}$ ***R1***

$\Rightarrow 1$ is even which is not true and so α cannot be an integer ***R1***

Note: Accept '$\Rightarrow 1$ is even which gives a contradiction'.

OR

$\frac{1}{2} = -\alpha^3 - 3\alpha$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow \left(-\alpha^3 - 3\alpha\right) \in \mathbb{Z}$ ***R1***

$-\alpha^3 - 3\alpha$ is is not an integer $\left(= \frac{1}{2}\right)$ and so α cannot be an integer
R1

Note: Accept '$-\alpha^3 - 3\alpha$ is not an integer $\left(= \frac{1}{2}\right)$ which gives a contradiction'.

OR

$\alpha = -\frac{1}{2(\alpha^2+3)}$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow -\frac{1}{2(\alpha^2+3)} \in \mathbb{Z}$ ***R1***

$-\frac{1}{2(\alpha^2+3)}$ is not an integer and so α cannot be an integer ***R1***

Note: Accept $-\frac{1}{2(\alpha^2+3)}$ is not an integer which gives a contradiction'.

THEN

so the equation $2x^3 + 6x + 1 = 0$ has no integer roots ***AG***

METHOD 2

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$ ***M1***

Note: Award ***M1*** for equivalent statements such as 'assume that α is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of x throughout the proof. Award ***M1*** for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$ and award subsequent marks based on this.

Note: Award ***M0*** for statements such as "let's consider the equation has integer roots…", "let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$…"

Note: Subsequent marks after this ***M1*** are independent of this ***M1*** and can be awarded.

let $f(x) = 2x^3 + 6x + 1$ (and $f(\alpha) = 0$)

$f'(x) = 6x^2 + 6 > 0$ for all $x \in \mathbb{R} \Rightarrow f$ is a (strictly) increasing function ***M1A1***

$f(0) = 1$ and $f(-1) = -7$ ***R1***

thus $f(x) = 0$ has only one real root between -1 and 0, which gives a contradiction

(or therefore, contradicting the assumption that $f(\alpha) = 0$ for some $\alpha \in \mathbb{Z}$), ***R1***

so the equation $2x^3 + 6x + 1 = 0$ has no integer roots ***AG***

[5 marks]

5. [Maximum mark: 7] 21M.2.AHL.TZ1.8

Consider the complex numbers $z = 2\left(\cos\frac{\pi}{5} + \mathrm{i}\sin\frac{\pi}{5}\right)$ and $w = 8\left(\cos\frac{2k\pi}{5} - \mathrm{i}\sin\frac{2k\pi}{5}\right)$, where $k \in \mathbb{Z}^{+}$.

(a) Find the modulus of zw. [1]

Markscheme

$(|zw| =)16$ ***A1***

[1 mark]

(b) Find the argument of zw in terms of k. [2]

Markscheme

attempt to find $\arg(z) + \arg(w)$ ***(M1)***

$\arg(zw) = \arg(z) + \arg(w)$

$= \frac{\pi}{5} - \frac{2k\pi}{5} \left(= \frac{(1-2k)\pi}{5}\right)$ ***A1***

[2 marks]

Suppose that $zw \in \mathbb{Z}$.

(c.i) Find the minimum value of k. [3]

Markscheme

$zw \in \mathbb{Z} \Rightarrow \arg(zw)$ is a multiple of π ***(M1)***

$\Rightarrow 1 - 2k$ is a multiple of 5 ***(M1)***

$k = 3$ ***A1***

[3 marks]

(c.ii) For the value of k found in part (i), find the value of zw. [1]

Markscheme

$zw = 16(\cos(-\pi) + \mathrm{i}\sin(-\pi))$

-16 ***A1***

[1 mark]

6. [Maximum mark: 7] 19N.1.AHL.TZ0.H_5

Consider the equation $z^4 = -4$, where $z \in \mathbb{C}$.

(a) Solve the equation, giving the solutions in the form $a + \mathrm{i}b$, where $a,\ b \in \mathbb{R}$. [5]

Markscheme

METHOD 1

$|z| = \sqrt[4]{4}\left(= \sqrt{2}\right)$ ***(A1)***

$\arg\left(z_1\right) = \frac{\pi}{4}$ ***(A1)***

first solution is $1 + \mathrm{i}$ ***A1***

valid attempt to find all roots (De Moivre or +/– their components) ***(M1)***

other solutions are $-1 + \mathrm{i}$, $-1 - \mathrm{i}$, $1 - \mathrm{i}$ ***A1***

METHOD 2

$z^4 = -4$

$(a + \mathrm{i}b)^4 = -4$

attempt to expand and equate **both** reals and imaginaries. ***(M1)***

$a^4 + 4a^3b\mathrm{i} - 6a^2b^2 - 4ab^3\mathrm{i} + b^4 = -4$

$\left(a^4 - 6a^4 + a^4 = -4 \Rightarrow\right)a = \pm 1$ ***and***
$\left(4a^3b - 4ab^3 = 0 \Rightarrow\right)a = \pm b$ ***(A1)***

first solution is $1 + \mathrm{i}$ ***A1***

valid attempt to find all roots (De Moivre or +/– their components) ***(M1)*** other solutions are $-1+\mathrm{i}$, $-1-\mathrm{i}$, $1-\mathrm{i}$ ***A1***

[5 marks]

(b) The solutions form the vertices of a polygon in the complex plane. Find the area of the polygon. [2]

Markscheme

complete method to find area of 'rectangle' ***(M1)***

$=4$ ***A1***

[2 marks]

7. [Maximum mark: 7] 23N.1.AHL.TZ1.6

Prove by mathematical induction that $5^{2n}-2^{3n}$ is divisible by 17 for all $n\in\mathbb{Z}^{+}$. [7]

Markscheme

base case $n=1:\ 5^{2}-2^{3}=25-8=17$ so true for $n=1$ ***A1***

assume true for $n=k$ i.e. $5^{2k}-2^{3k}=17s$ for $s\in\mathbb{Z}$ OR $5^{2k}-2^{3k}$ is divisible by 17 ***M1***

Note: The assumption of truth must be clear. Do not award the ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks can still be awarded.

EITHER

consider $n = k + 1$: *M1*

$5^{2(k+1)} - 2^{3(k+1)}$

$= \left(5^2\right)5^{2k} - \left(2^3\right)2^{3k}$ *A1*

$= (25)5^{2k} - (8)2^{3k}$

$= (17)5^{2k} + (8)5^{2k} - (8)2^{3k}$ OR $(25)5^{2k} - (25)2^{3k} + (17)2^{3k}$

A1

$= (17)5^{2k} + 8\left(5^{2k} - 2^{3k}\right)$ OR $25\left(5^{2k} - 2^{3k}\right) + (17)2^{3k}$

$= (17)5^{2k} + 8(17s)$ OR $25(17s) + (17)2^{3k}$

$= 17\left(5^{2k} + 8s\right)$ OR $17\left(25s + 2^{3k}\right)$ which is divisible by 17

A1

OR

$\left(5^{2k} - 2^{3k}\right) \times 5^2 = 5^{2k+2} - 25 \times 2^{3k} = 17s \times 25$ *M1*

$= 5^{2k+2} - 8 \times 2^{3k} - 17 \times 2^{3k} = 17s \times 25$ *A1*

$= 5^{2k+2} - 2^{3k+3} - 17 \times 2^{3k} = 17s \times 25$

$= 5^{2(k+1)} - 2^{3(k+1)} - 17 \times 2^{3k} = 17s \times 25$ *A1*

$= 5^{2(k+1)} - 2^{3(k+1)} = 17s \times 25 + 17 \times 2^{3k}$

hence for $n = k + 1\ :\ 5^{2(k+1)} - 2^{3(k+1)} = 17\left(25s + 2^{3k}\right)$ is divisible by 17 *A1*

THEN

since true for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, therefore true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Only award **R1** if 4 of the previous 6 marks have been awarded
Note: 5^{2k} and 2^{3k} may be replaced by 25^k and 8^k throughout.

[7 marks]