CHAPTER THREE: QUADRATICS

Review November 20 ∽ Test December 2

The most common functions in math at this level are quadratic functions, whose graphs are parabolas. Important in this chapter and in future math is the relationship between x-intercepts, zeros, roots, and solutions. Roughly speaking, all of these refer to values which make y equal zero. For quadratic equations, they can be found by graphing, factoring, completing the square, or using the quadratic formula. Sometimes they are imaginary rather than real, such as in the equation $x^2 = -1$ which has as solutions the imaginary numbers $x = \pm\sqrt{-1}$. This happens when the parabola is completely above or below the x-axis and never crosses it, resulting in no x-intercepts.

3-A Graphs of Quadratic Functions — Wednesday • 11/4

standard form • vertex form • vertex • axis of symmetry • maximum • minimum

1. Identify the vertex of a quadratic in vertex form.
2. Write a quadratic equation in standard form.
3. Identify the vertex of a parabola in standard form.
4. Identify the vertex of a parabola in intercept form.
5. Sketch a parabola and its axis of symmetry.

3-B Simplifying Radical Expressions — Friday • 11/6

radicand • radical

1. Simplify a square root.
2. Simplify an n^{th} root.
3. Rationalize a denominator with one or two terms.

3-C Complex Numbers — Tuesday • 11/10

imaginary number • complex number • complex plane • complex conjugate

1. Write a fractional complex number in standard form.
2. Plot numbers on the complex plane.
3. Solve equations with complex solutions.
4. Add, subtract, and multiply complex numbers.
5. Divide complex numbers.

3-D Factoring — Friday • 11/13

factoring • common monomial

1. Factor a trinomial by guessing and checking.
2. Factor a common monomial out of each term of a polynomial.
3. Factor a polynomial by grouping.
4. Factor $ax^2 + bx + c$ by grouping.
5. Factor a perfect square trinomial, a difference of two squares, a difference of two cubes, or a sum of two cubes.

3-E Solving Quadratic Equations

Wednesday • 11/18

completing the square • quadratic formula • solution • zero • root • x-intercept

1. Solve a quadratic equation by factoring.
2. Solve a quadratic equation by isolating a square.
3. Solve a quadratic equation by completing the square.
4. Solve a quadratic equation with the quadratic formula.
5. Solve any quadratic equation.
6. Find the solutions, zeros, roots, or x-intercepts of a quadratic.

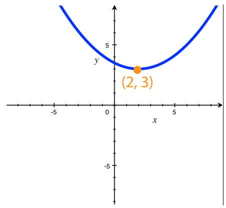
3-A Graphs of Quadratic Equations

VERTEX Form of a Quadratic Equation is $y = a(x - h)^2 + k$. The VERTEX (h, k) is the tip of the parabola.

❶ Identify the vertex of a parabola in vertex form.

1. Identify h and k.
2. The vertex is (h, k).

❶ Identify the vertex.



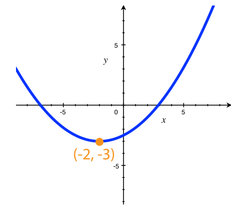
a) $y = \frac{1}{8}(x - 2)^2 + 3$

1. $h = 2$, $k = 3$
2. $(2, 3)$

b) $y = \frac{1}{8}(x + 2)^2 - 3$

$h = -2$, $k = -3$

$(-2, -3)$



STANDARD Form of a Quadratic Equation is $y = ax^2 + bx + c$ (or $ax^2 + bx + c = 0$).

❷ Write a quadratic equation in standard form.

1. Get y or zero by itself on one side.
2. Multiply as needed to remove parentheses.
3. Combine like terms, putting the combined terms in order from highest exponent to lowest.

❷ $2x^2 = 3(x + 5)^2 + 100$

1. $3(x + 5)^2 + 100 - 2x^2 = 0$
2. $3(x^2 + 10x + 25) + 100 - 2x^2 = 0$

   $3x^2 + 30x + 75 + 100 - 2x^2 = 0$
3. $x^2 + 30x + 175 = 0$

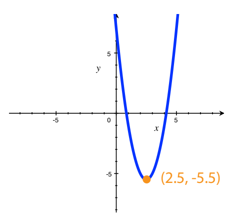
For a quadratic equation in standard form $f(x) = ax^2 + bx + c$, $h = \frac{-b}{2a}$ and $k = f(h)$.

❸ Identify the vertex of a parabola in standard form.

1. Identify a and b.
2. Calculate $h = -b \div 2a$.
3. Plug h into the equation to find k.
4. The vertex is (h, k).

❸ $f(x) = 2x^2 - 10x + 7$

1. $a = 2$, $b = -10$
2. $h = \frac{-(-10)}{2(2)} = 2.5$
3. $k = f(2.5) = 2(2.5)^2 - 10(2.5) + 7 = -5.5$
4. The vertex is $(2.5, -5.5)$.



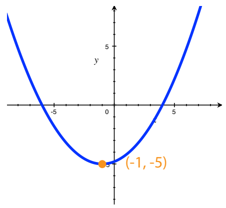
INTERCEPT Form of a Quadratic Equation is $y = a(x - p)(x - q)$. The x-intercepts are $(p, 0)$ and $(q, 0)$, and the x-coordinate of the vertex is midway between these: $h = \frac{p+q}{2}$. As always, $k = f(h)$.

❹ Identify the vertex of a parabola in intercept form.

1. Identify p and q.
2. Calculate $h = (p + q) \div 2$.
3. Plug h into the equation to find k.
4. The vertex is (h, k).

❹ $f(x) = \frac{1}{5}(x - 4)(x + 6)$

1. $p = 4$, $q = -6$
2. $h = \frac{4-6}{2} = -1$
3. $k = \frac{1}{5}(-1 - 4)(-1 + 6) = -5$
4. The vertex is $(-1, -5)$.



The AXIS OF SYMMETRY of a Parabola is the line $x = h$ that cuts the parabola exactly in half.

If a is positive, the parabola opens upward and k is the MINIMUM of the quadratic.

If a is negative, the parabola opens downward and k is the MAXIMUM of the quadratic.

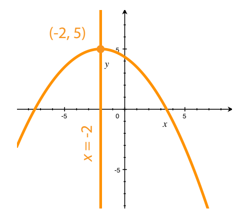
The closer a is to 0, the wider the parabola is.

❺ Sketch a parabola and its axis of symmetry.

1. Find and plot the vertex (h, k) (see ❶, ❸, or ❹).
2. Draw the axis of symmetry, which is the vertical line $x = h$ through the vertex.
3. Use the value of a to determine the direction and approximate shape of the parabola.

❺ $y = \frac{-1}{6}(x + 2)^2 + 5$

1. The vertex is $(-2, 5)$.
2. The axis of symmetry is $x = -2$.
3. $a = \frac{-1}{6}$ is negative and close to zero, so the parabola opens down and is fairly wide.



3-B Simplifying Radical Expressions

An expression under a root sign is a RADICAND. Together with the root sign, it is called a RADICAL.

$a^{1/n}$ means $\sqrt[n]{a}$ Therefore, the properties of powers listed in section 1-D can be applied to roots as well. In particular, by the product of powers property, $\sqrt[n]{ab} = \sqrt[n]{a} \bullet \sqrt[n]{b}$.

❶ Simplify a square root.

1. Use the product property to separate the radical into two radicals, the first of which is the square root of a square. Keep in mind that a variable with an even exponent can be written as a square: $x^{2a} = (x^a)^2$.
2. Take the square root of the first radical.
3. Repeat steps 1 and 2 with the remaining radical, if possible.
4. Simplify if needed.

❶ $\sqrt{75x^2y^6z^7}$

1. $\sqrt{25x^2y^6z^6}\sqrt{3z}$
2. $5xy^3z^3\,\sqrt{3z}$

❷ Simplify an n^{th} root.

1. Do steps 1-4 above, except use n^{th} roots instead of square roots. Keep in mind that a variable with an exponent divisible by n can be written as an n^{th} power: $x^{na} = (x^a)^n$.

❷ $\sqrt[3]{800x^3y^6z^8}$

2. $\sqrt[3]{8x^3y^6z^6}\,\sqrt[3]{100z^2}$
3. $2xy^2z^2\sqrt[3]{100z^2}$

The CONJUGATE of a number $a + \sqrt{b}$ is $a - \sqrt{b}$.

Multiplying a number by its conjugate results in a rational (nonradical) number: $(a + \sqrt{b})\,(a - \sqrt{b}) = a^2 + a\sqrt{b} - a\sqrt{b} - (\sqrt{b})^2 = a^2 - b$.

A denominator with a radical is not considered simplified. To RATIONALIZE a Denominator is to rewrite a fraction so that there is no radical in the denominator.

❸ Rationalize a denominator with one or two terms.

1. Multiply the numerator and denominator by the denominator if it is a single term, or by the conjugate of the denominator if it has two terms.
2. Simplify the radical.
3. Reduce.

❸ Simplify.

a) $\frac{12}{\sqrt{20}}$

1. $\frac{12}{\sqrt{20}} \bullet \frac{\sqrt{20}}{\sqrt{20}} = \frac{12\sqrt{20}}{20}$
2. $\frac{12(2\sqrt{5})}{20} = \frac{24\sqrt{5}}{20}$
3. $\frac{6\sqrt{5}}{5}$

b) $\frac{12}{8-\sqrt{20}}$

$\frac{12}{8-\sqrt{20}} \bullet \frac{8+\sqrt{20}}{8+\sqrt{20}} = \frac{96+12\sqrt{20}}{8^2-20}$

$\frac{96+12(2\sqrt{5})}{44} = \frac{96+24\sqrt{5}}{44}$

$\frac{24+6\sqrt{5}}{11}$

3-C Complex Numbers

A number involving the square root of a negative is IMAGINARY (Nonreal). The Imaginary Unit i is defined as $i = \sqrt{-1}$.

A COMPLEX Number is a number that can be written in the form $a + bi$, where a and b are real numbers. Since b can be zero, all numbers can be considered complex.

Standard form of a complex number is $a + bi$.

❶ Write a fractional complex number in standard form.

1. Split it into its real part and its imaginary part, each with the same denominator.

❶ Write $\frac{-10 + 14i}{9}$ in standard form.

$\frac{-10}{9} + \frac{14}{9}i$

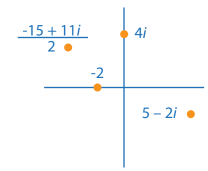
A complex number $a + bi$ can be represented on the Complex PLANE by the coordinates (a, b).

❷ Plot numbers on the complex plane.

1. Write the number as $a + bi$. For real numbers, $b = 0$. For fractions, see ❶.
2. Plot the point (a, b).

❷ Plot the following on the complex plane.

a) $5 - 2i$	b) -2	c) $4i$	d) $\frac{-15 + 11i}{2}$
1. $5 - 2i$	$-2 + 0i$	$0 + 4i$	$\frac{-15}{2} + \frac{11}{2}i$
2. $(5, -2)$	$(-2, 0)$	$(0, 4)$	$(\frac{-15}{2}, \frac{11}{2})$



❸ Solve equations with complex solutions.

1. Solve the equation normally, but change any number $\sqrt{-c}$ to $i\sqrt{c}$.

❸ $2x^2 + 91 = 3$

$x^2 = -44$

$x = \pm\sqrt{-44}$

$x = \pm i\sqrt{44}$

$x = \pm 2i\sqrt{11}$

❹ Add, subtract, and multiply complex numbers.

1. Treat i like a variable, but change i^2 to -1.

❹ Calculate.

a) $(5 + 2i) - (8 - 3i)$

$5 + 2i - 8 + 3i$

$-3 + 5i$

b) $(5 + 2i)(8 - 3i)$

$40 + 16i - 15i - 6i^2$

$40 + i - 6(-1) = 46 + i$

The Complex Conjugate of a number $a + bi$ is $a - bi$.

Multiplying a number by its complex conjugate results in a real number: $(a + bi)(a - bi) = a^2 + abi - abi - (bi)^2 = a^2 + b^2$.

❺ Divide complex numbers.

1. Multiply the numerator and denominator by the complex conjugate of the denominator.
2. Convert i^2 to -1, and combine like terms.
3. Put into $a + bi$ form, and reduce.

❺ $\frac{3+4i}{2-10i}$

1. $\frac{3+4i}{2-10i} \bullet \frac{2+10i}{2+10i} = \frac{6+30i+8i+40i^2}{2^2+10^2}$
2. $\frac{6+38i-40}{2^2+10^2} = \frac{-34+38i}{104}$
3. $\frac{-34}{104} + \frac{38}{104}i = \frac{-17}{52} + \frac{19}{52}i$

3-D Factoring

Just as a factor of a whole number is a whole number that divides evenly into it, a FACTOR of a Polynomial is a polynomial that divides evenly into it. For example, 2 and 7 are factors of 14, and $(x + 5)$ and $(x - 5)$ are factors of $x^2 - 25$. Factoring a polynomial is writing it as a product of other polynomials.

❶ Factor a trinomial by guessing and checking.

1. Choose two terms that multiply together to equal the first term in the trinomial.
2. Choose two terms that multiply together to equal the last term in the trinomial.
3. Create two binomials from your four terms.
4. Multiply the binomials together to see if you get the desired polynomial. If not, start over with new terms.

❶ $12x^2 - 4x - 5$	retry:
1. $6x$ and $2x$	$6x$ and $2x$
2. 5 and -1	-5 and 1
3. $(6x + 5)$ and $(2x - 1)$	$(6x - 5)$ and $(2x + 1)$
4. $(6x + 5)(2x - 1) = 12x^2 + 4x + 5$ X	$(6x - 5)(2x + 1) = 12x^2 - 4x - 5$ √

A COMMON Monomial is a term that divides evenly into every term in a polynomial.

❷ Factor a common monomial out of each term of a polynomial.

1. Find a monomial that divides evenly into every term in the polynomial.
2. Divide every term by the common monomial.
3. Write the common monomial, followed by the new polynomial in parentheses.
4. Repeat if possible.

❷ $40x^5 - 8x^3 + 20x^2$

1. $4x^2$ divides into all three terms.
2. $(40x^5 - 8x^3 + 20x^2) \div 4x^2 = 10x^3 - 2x + 5$
3. $4x^2\,(10x^3 - 2x + 5)$

Sometimes one monomial is common to some of the terms in a polynomial, and another monomial is common to the other terms. If factoring these separate monomials out of their respective terms leaves the same quotient each time, this grouping can help in factoring the polynomial.

❸ Factor a polynomial by grouping.

1. Sort the terms of the polynomial into groups such that each group has its own common monomial.
2. Factor a common monomial out of each group.
3. If the quotient of each group is the same, the factors of the polynomial are this quotient and the sum of the common monomials. If not, start over with different groupings or different monomials.

❸ $2x^3 - 8x^2 + 5x - 20$

1. $(2x^3 - 8x^2) + (5x - 20)$
2. $2x^2(x - 4) + 5(x - 4)$
3. $(2x^2 + 5)(x - 4)$

A factorable trinomial $ax^2 + bx + c$ can be factored by grouping after splitting up bx into $b_1x + b_2x$, where $b_1b_2 = ac$.

❹ Factor $ax^2 + bx + c$ by grouping.

1. Calculate ac.
2. Identify a pair of factors of ac whose sum is b.
3. Using this pair of factors, rewrite bx as the sum of two separate terms.
4. Factor by grouping (see ❸).

❹ $12x^2 - 4x - 5$

1. $ac = 12(-5) = -60$
2. $-10(6) = -60$, and $-10 + 6 = -4$
3. $12x^2 - 10x + 6x - 5$
4. $2x(6x - 5) + 1(6x - 5)$

   $(2x + 1)(6x - 5)$

The following special cases are easy to factor when recognized.

Type of Polynomial	Polynomial	Factors
Perfect Square Trinomial	$a^2 + 2ab + b^2$	$(a + b)^2$
Difference of Two Squares	$a^2 - b^2$	$(a + b)\ (a - b)$
Difference of Two Cubes	$a^3 - b^3$	$(a - b)\ (a^2 + ab + b^2)$
Sum of Two Cubes	$a^3 + b^3$	$(a + b)\ (a^2 - ab + b^2)$

❺ Factor a perfect square trinomial, a difference of two squares, a difference of two cubes, or a sum of two cubes.

1. Disregarding negatives, let a be the square root or cube root of the first term and let b be the square root or cube root of the last term.
2. See which of the four polynomials in the second column above matches the polynomial you are factoring.
3. Fill in a and b in the factorization shown in the third column above.
4. For a possible perfect square, give b the same sign as the middle term of the polynomial, and multiply the factors together to see if they do in fact result in the desired polynomial.

❺ Use special cases to factor the following polynomials if possible.

Polynomial	a	b	Type of Polynomial	Factors
a) $x^2 - 100y^2$	x	$10y$	difference of two squares	$(x + 10y)\ (x - 10y)$
b) $x^{10} - 100y^4$	x^5	$10y^2$	difference of two squares	$(x^5 + 10y^2)\ (x^5 - 10y^2)$
c) $x^3 - 1000y^3$	x	$10y$	difference of two cubes	$(x - 10y)\ (x^2 - 10xy + 100y^2)$
d) $27m^3 + 8p^3$	$3m$	$2p$	sum of two cubes	$(3m + 2p)\ (9m^2 - 6mp + 4p^2)$
e) $4x^2 - 20x + 25$	$2x$	-5	perfect square	$(2x - 5)^2$
f) $4x^2 - 10x + 25$	$2x$	-5	not a perfect square, because $(2x - 5)^2 = 4x^2 - 20x + 25$, not $4x^2 - 10x + 25$	

❻ Factor any polynomial.

1. If possible, factor out a common monomial from every term (see ❷).
2. If possible, factor a perfect square, a difference of squares or of cubes, or a sum of cubes (see ❹).
3. If the polynomial is not yet completely factored, try factoring by grouping (see ❸) or by guessing and checking (see ❶).

❻ $80x^7 - 180x^5$

1. $20x^5\ (4x^2 - 9)$
2. $20x^5\ (2x + 3)\ (2x - 3)$

3-E Solving Quadratic Equations

If $ab = 0$, then $a = 0$ or $b = 0$.

❶ Solve a quadratic equation by factoring.

1. Isolate zero on one side of the equation.
2. Factor.
3. Set each factor equal to zero.
4. Solve each equation.

❶ $18x^3 = 50x$

1. $18x^3 - 50x = 0$
2. $2x\,(9x^2 - 25) = 0$

   $2x\,(3x + 5)\,(3x - 5) = 0$
3. $2x = 0$ $\quad 3x + 5 = 0$ $\quad 3x - 5 = 0$
4. $x = 0$ $\quad x = \frac{-5}{3}$ $\quad x = \frac{5}{3}$

If one side of an equation is a square and the other side is a constant, the equation can be solved by taking the square root of each side.

❷ Solve a quadratic equation by isolating a square.

1. Rewrite the equation so that there is a squared polynomial alone on one side and a constant alone on the other.
2. Take the square root of each side.
3. Solve the + equation.
4. Solve the - equation.

❷ $8(2x + 3)^2 - 80 = 120$

1. $8(2x + 3)^2 = 200$

   $(2x + 3)^2 = 25$
2. $2x + 3 = \pm 5$
3. $2x + 3 = 5$

   $x = 1$
4. $2x + 3 = -5$

   $x = -4$

If there isn't already a squared polynomial in a quadratic equation, you can make one by COMPLETING THE SQUARE.

❸ Solve a quadratic equation by completing the square.

1. Subtract c from each side.
2. Divide each term by a.
3. Complete the square by adding $(\frac{b}{2})^2$ to each side.
4. Rewrite the square side as $(x + \frac{b}{2})^2$.
5. Square root each side.
6. Solve the + equation.
7. Solve the - equation.

❸ $2x^2 + 20x + 34 = 16$

1. $2x^2 + 20x = -18$
2. $x^2 + 10x = -9$
3. $x^2 + 10x + 25 = -9 + 25$
4. $(x + 5)^2 = 16$
5. $x + 5 = \pm 4$
6. $x + 5 = 4$

   $x = -1$

   $x + 5 = -4$

   $x = -9$

Any quadratic equation in standard form can be solved by plugging a, b, and c into the QUADRATIC FORMULA: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

❹ Solve a quadratic equation with the quadratic formula.

1. Put the equation in standard form.
2. Identify a, b, and c.
3. Plug a, b, and c into the quadratic formula.
4. Simplify.

❹ $12x^2 + 17x = 7$

1. $12x^2 + 17x - 7 = 0$
2. $a = 12, b = 17, c = -7$
3. $x = \frac{-17 \pm \sqrt{17^2 - 4(12)(-7)}}{2(12)}$
4. $x = \frac{-17 \pm 25}{24} = \frac{1}{3}$ or $\frac{-7}{4}$

❺ Solve any quadratic equation.

1. If you can easily get a square by itself on one side and a constant by itself on the other, do so and then square root each side. See ❷ and ❸. In general, completing the square is easy if b is divisible by $2a$.
2. Otherwise put the equation in standard form and do one of the following:

a) Factor the expression, set each factor equal to zero, and solve (see ❶).

b) Use the quadratic formula (see ❹).

❺ $x^2 + 10x + 30 = 9$

1. $x^2 + 10x + 25 = 4$

$(x + 5)^2 = 4$

$x + 5 = \pm 2$

$x = -7$ or -3

2. $x^2 + 10x + 21 = 0$

a) $(x + 7)(x + 3) = 0$

$x + 7 = 0$ or $x + 3 = 0$

$x = -7$ or $x = -3$

b) $x = \frac{-10 \pm \sqrt{10^2 - 4(1)(21)}}{2(1)}$

$x = \frac{-10 \pm 4}{2} = \frac{-14}{2}$ or $\frac{-6}{2} = -7$ or -3

A SOLUTION is a value that makes an equation true.

A ZERO or ROOT is a value that makes an expression or function equal to zero.

An X-INTERCEPT is a point $(x, 0)$ on a graph, where x is a real zero.

The DISCRIMINANT of a Quadratic Equation is the radicand $b^2 - 4ac$. If this is positive, the quadratic equation has two real solutions and thus two x-intercepts. If the discriminant is zero, the equation has one real solution, and if it is negative, there are no real solutions.

❻ Find the solutions, zeros, roots, or x-intercepts of a quadratic.

1. If the expression has no equals sign or is equal to y, make it equal to zero.
2. Solve the equation to find the solutions, zeros, or roots.
3. The x-intercepts are the same values as the zeros, unless the zeros are imaginary, in which case there are no x-intercepts.

❻

The solutions to $x^2 + 10x + 21 = 0$ are $x = -7$ and -3.

The zeros or roots of $f(x) = x^2 + 10x + 21$ are $x = -7$ and -3.

The x-intercepts of $f(x) = x^2 + 10x + 21$ are $(-7, 0)$ and $(-3, 0)$.