CHAPTER TWO: FUNCTIONS

Test: Thursday, October 3

Functions are the basic building blocks of algebra and most further math. Functions have a domain, a range, and an inverse, and they can be composed, sketched, and transformed. There are many different families of functions, a few key ones of which are explored in this chapter.

2-A Composition and Inverses — Friday • 9/6

composition • inverse • one-to-one • equation

❶ Evaluate compositions of functions.
❷ Find a simplified expression for a composition of functions.
❸ Identify the inverse of a basic function by definition.
❹ Identify the inverse of a function conceptually.
❺ Find the inverse of a relation algebraically.
❻ Find the inverse of a relation graphically.
❼ Determine whether or not the inverse of a graph is a function.

2-B Power and Root Functions — Wednesday • 9/11

n^{th} root • extraneous solution

❶ Find real n^{th} roots of a number a with a calculator.
❷ Evaluate $\sqrt[n]{a^m}$ or $a^{m/n}$ by hand.
❸ Solve a power equation or root equation.

2-C Exponential Functions — Friday • 9/13

exponential • growth factor • decay factor • compound interest

❶ Identify a scenario's rate of increase or decrease and its growth or decay factor.
❷ Write a function for an exponential growth or decay situation, and use it to calculate future values.
❸ Calculate an account balance with compounded annual interest.

2-D Logarithmic Functions — Tuesday • 9/17

logarithm • common log • natural log • change of base property • exponentiate • half-life

❶ Simplify the composition of a logarithmic and an exponential function.
❷ Evaluate a simple logarithm by hand.
❸ Evaluate a common or natural logarithm with a calculator.
❹ Simplify a logarithmic expression.
❺ Simplify a logarithm in which the base and argument are both powers of the same base.
❻ Evaluate any logarithm.
❼ Solve a basic exponential equation.
❽ Solve an equation in which one exponential expression is equal to another.
❾ Solve a logarithmic equation.
❿ Translate a description of a half-life situation into an equation, and solve it.
⓫ Determine half-life based on decay rate or decay rate based on half-life.

2-E Sketches of Functions — Monday • 9/23

asymptote

❶ Sketch a power function $f(x) = x^n$.
❷ Sketch a root function $f(x) = \sqrt[n]{x}$.
❸ Sketch an increasing exponential function $f(x) = b^x$.
❹ Sketch a decreasing exponential function $f(x) = b^x$.
❺ Sketch an increasing logarithmic function $f(x) = \log_b x$.
❻ Sketch a decreasing logarithmic function $f(x) = \log_b x$.

2-F Transformations

Tuesday • 9/24

transformation • pre-image • image • translation • stretch • reflection

❶ Translate a function h units right and k units up.

❷ Stretch a function horizontally by a factor of b and vertically by a factor of a.

❸ Reflect a function across the y-axis and/or the across the x-axis.

❹ Apply multiple transformations to a function.

❺ Given the graph of $f(x)$, sketch $f(x-h)+k$.

❻ Given the graph of $f(x)$, sketch $a \cdot f\left(\frac{x}{b}\right)$.

❼ Given the graph of $f(x)$, sketch $f(-x)$ or $-f(x)$.

❽ Use the equation of a pre-image to find the equation of a graph.

2-A Composition and Inverses

COMPOSITION of Functions is using an entire function or its value as the input for a function.

The composition of the function f with the function g is written $f(g(x))$ or $(f \circ g)(x)$.

❶ Evaluate compositions of functions.

1. Calculate the value of the inner or last function.
2. Repeat step 1 for each additional function.

❶ Given $f(x) = 4x - 10$ and $g(x) = x^2 + 2x - 3$, evaluate the following.

	a) $f(g(3))$	b) $g(f(3))$	c) $f(g(f(3)))$
1.	$= f(3^2 + 2(3) - 3)$	$= g(4(3) - 10)$	$= f(g(4(3) - 10)$
	$= f(12)$	$= g(2)$	$= f(g(2))$
2.	$= 4(12) - 10 = 38$	$= 2^2 + 2(2) - 3 = 5$	$= f(5)$
			$= 4(5) - 10 = 10$

❷ Find a simplified expression for a composition of functions.

1. Substitute the inner or last function's expression, in parentheses, for each variable in the next function.
2. Simplify.
3. Repeat steps 1 and 2 for each additional function.

❷ Using the functions f and g, above, give an expression for the following.

	a) $f(g(x))$	b) $g(f(x))$	c) $f(g(f(x)))$
1.	$= f(x^2 + 2x - 3)$	$= g(4x - 10)$	$= f(g(4x - 10))$
	$= 4(x^2 + 2x - 3) - 10$	$= (4x - 10)^2 + 2(4x - 10) - 3$	$= f((4x - 10)^2 + 2(4x - 10) - 3$
2.	$= 4x^2 + 8x - 22$	$= 16x^2 - 72x + 77$	$= f(16x^2 - 72x + 77)$
3.			$= 4(16x^2 - 72x + 77) - 10$
			$= 64x^2 - 288x + 298$

The INVERSE of a Relation is the relation that cancels the original relation.

If a function f has an inverse that is also a function, then f is ONE-TO-ONE and its inverse is labeled f^{-1}. $f^{-1}(f(x)) = x$

❸ Identify the inverse of a basic function by definition.

The inverse of addition is subtraction.

The inverse of multiplication is division.

The inverse of a power is a root.

The inverse of an exponential is a logarithm.

❸ Identify the inverse of each of the following functions, and verify that $f^{-1}(f(x)) = x$.

a) $a(x) = x + 5$	$a^{-1}(x) = x - 5$	$a^{-1}(a(x)) = x + 5 - 5 = x$
b) $b(x) = 5x$	$b^{-1}(x) = \frac{x}{5}$	$b^{-1}(b(x)) = \frac{5x}{5} = x$
c) $c(x) = x^5$	$c^{-1}(x) = \sqrt[5]{x}$	$c^{-1}(c(x)) = \sqrt[5]{x^5} = x$
d) $d(x) = 5^x$	$d^{-1}(x) = \log_5 x$	$d^{-1}(d(x)) = \log_5 5^x = x$

The inverse of a relation can be found by switching the independent variable with the dependent variable.

❹ Identify the inverse of a function conceptually.

1. Switch the variables so that the independent variable becomes the dependent variable and vice versa.

❹ $f(x) = 7x$ is the number of minutes it takes Rex to run x miles. (Miles are plugged in to calculate minutes.)

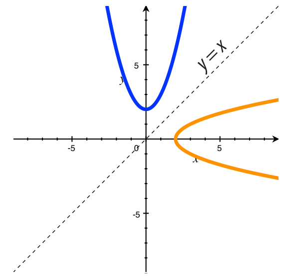
1. $f^{-1}(x) = \frac{x}{7}$ is the number of miles Rex runs in x minutes. (Minutes are plugged in to calculate miles.)

❺ Find the inverse of a relation algebraically.

1. Switch the independent variable with the dependent variable. y can be written instead of $f(x)$.
2. Solve for the new dependent variable.
3. If the new equation is a function, write the dependent variable using $f^{-1}(x)$ notation.

❺ Find the inverse of the following functions.

	a) $f(x) = x - 8$	b) $g(x) = 2x + 1$	c) $h(x) = x^2$
1.	$x = y - 8$	$x = 2y + 1$	$x = y^2$
2.	$y = x + 8$	$y = \frac{x}{2} - \frac{1}{2}$	$y = \pm\sqrt{x}$
3.	$f^{-1}(x) = x + 8$	$g^{-1}(x) = \frac{x}{2} - \frac{1}{2}$	



The inverse of a graph can be seen by reversing the coordinates of each point on the graph, such as (1, 3) changing to (3, 1). This results in reflecting the graph of the original relation across the $y = x$ diagonal.

❻ Find the inverse of a relation graphically.

1. Draw the line $y = x$.

2. Reflect the graph of the original relation across this line.

❻ Sketch the graph of the inverse of the parabola shown at right.

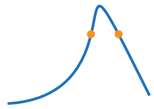
If a graph has two different points with the same y value, then its inverse is not a function.

❼ Identify whether or not the inverse of a graph represents a function.

1. It is a function unless there is a point on the graph that is directly to the right of another point on the graph.

❼ a) inverse is a function

b) inverse is not a function



2-B Power and Root Functions

An n^{th} ROOT of a value a is a number x such that $x^n = a$. It can be written $\sqrt[n]{a}$ or $a^{1/n}$.

For $a \neq 0$, there are n n^{th} roots of a, of which the notation $\sqrt[n]{a}$ (or $a^{1/n}$) represents a specific one based on the values of a and n:

- If a is positive, $\sqrt[n]{a}$ is the positive real n^{th} root of a. If n is even, there is also a negative real n^{th} root of a, represented as $-\sqrt[n]{a}$.
- If a is negative and n is odd, $\sqrt[n]{a}$ is the only real n^{th} root of a, and it is negative.
- If a is negative and n is even, there are no real n^{th} roots of a, and $\sqrt[n]{a}$ has no real value.

❶ Find real n^{th} roots of a number a with a calculator.

1. If a negative and n is even, there are no real roots.
2. Otherwise, type `a^(1/n)`. Put $\pm$ if n is even.

❶ Find all real roots, if any.

a) 5^{th} roots of -50
`(-50)^(1/5)` $\approx$ -2.19

b) 6^{th} roots of 50
$\pm$`50^(1/6)` $\approx \pm 1.92$

c) 6^{th} roots of -50
-50 is negative and 6 is even, so no real roots

❷ Evaluate $\sqrt[n]{a^m}$ or $a^{m/n}$ by hand.

1. Write it as $(\sqrt[n]{a})^m$
2. Evaluate $\sqrt[n]{a}$.
3. Put this value to the power of m.

❷ $8^{5/3}$

1. $(\sqrt[3]{8})^5$
2. 2^5
3. 32

Equations in which an n^{th} power can be isolated can be solved by taking the n^{th} root on each side, or vice versa.

An EXTRANEOUS Solution is one which does not work in the original equation despite the algebraic steps all being done correctly. This will occur when an impossible equation arises, such as an even root equaling a negative number.

❸ Solve a power equation or root equation.

1. Isolate the power or root.
2. Cancel the n^{th} power by taking the n^{th} root on each side, or cancel the n^{th} root by taking the n^{th} power on each side. When taking an even root, put $\pm$.
3. Continue solving.
4. Plug the solution in to the original equation to check it.

❸ Solve.

a) $2(x+5)^4 - 30 = 66$

1. $2(x+5)^4 = 96$
$(x+5)^4 = 48$
2. $x+5 = \pm 48^{1/4}$
$x+5 \approx \pm 2.63$
3. $x+5 \approx 2.63$ | $x+5 \approx -2.63$
$x \approx -2.37$ | $x \approx -7.63$
4. $2(-2.37+5)^4 - 30 \approx 66$ ✓ | $2(-7.63+5)^4 - 30 \approx 66$ ✓

b) $2(\sqrt[4]{x+5}) + 92 = 80$

$2(\sqrt[4]{x+5}) = -12$
$\sqrt[4]{x+5} = -6$
$x+5 = (-6)^4$
$x+5 = 1296$
$x = 1291$
$2(\sqrt{1291+5}) + 92 = 104 \neq 80$ ✕
$x = 1291$ is an extraneous solution.

2-C Exponential Functions

An EXPONENTIAL Function is one in which the independent variable is in the exponent: $f(x) = b^x$.

In Exponential GROWTH, $b = 1 + r$ is called the GROWTH FACTOR, and r is the rate of increase. If r is negative, b will be less than one and $-r$ is the rate of decrease; this is called Exponential DECAY.

An increase is an addition, a decrease is a subtraction, and a factor is a multiplier. Therefore, if a value is increasing by x or is x times more, then x is r; if a value is decreasing by x, then x is $-r$; and if a value is x times as much, then x is b.

❶ Identify a scenario's rate of increase or decrease and its growth or decay factor.

1. Change the given value to a decimal if needed.
2. If the change given is a decrease of x, then $r = -x$ and $b = 1 + r$.
   If the change given is an increase of x, then $r = x$ and $b = 1 + r$.
   If the change given is a factor of x, then $b = x$ and $r = b - 1$.

❶ a) 4% less
$r = -.04$
$b = 1 + -.04 = 0.96$

b) 200% more
$r = 2$
$b = 2 + 1 = 3$

c) 3 times as much
$b = 3$
$r = 3 - 1 = 2$

d) 3 times more
$r = 3$
$b = 3 + 1 = 4$

$f(x) = ab^x$ represents a quantity $f(x)$ at time x, given a growth rate of $r = b - 1$ and an initial quantity of a at time $x = 0$.

❷ Write a function for an exponential growth or decay situation, and use it to calculate future values.

1. Identify the initial amount a.
2. Identify the growth or decay factor b.
3. Fill in a and b in the formula $f(x) = ab^x$.
4. Identify the units for the independent variable x.
5. Determine x, the amount of time past the starting point. x will be negative for times prior to the starting point.
6. Calculate $f(x)$.

❷ An account earns 4.90% annual interest and has $8330 in 2024. Calculate the amount in the account in the following years.

a) 2031 b) 2020

1. $a = 8330$
2. $b = 1 + .049 = 1.049$
3. $f(x) = 8330(1.049)^x$
4. x = # of years after 2020
5. $x = 2030 - 2023 = 7$ $\quad$ $x = 2020 - 2024 = -4$
6. $f(8) = 8330(1.049)^7 = \$11,643$ $\quad$ $f(-3) = 8330(1.049)^{-4} = \$6,879$

When going back in time, avoid the mistake of making the growth rate r negative instead of making the amount of time x negative: $a(1 + r)^{-x} \neq a(1 - r)^x$

COMPOUND INTEREST is a percentage of a balance added periodically to the balance, such that the amount added increases each time because it is a percentage of a bigger balance. It is exponential growth, but it is typically written using the notation $A = P\left(1 + \frac{r}{n}\right)^{nt}$ rather than $f(x) = ab^x$. A is the ending amount, P is the principal (initial amount), r is the annual interest rate (rate of increase), n is the number of compoundings per year, and t is the number of years.

When an interest rate is given, it is always an annual rate unless stated otherwise, even if it is compounded more than once per year.

❸ Calculate an account balance with compounded annual interest.

1. Identify the initial amount P.
2. Identify the annual interest rate r, and write it as a decimal.
3. Identify the number of times, n, that interest is compounded per year.
4. Identify the number of years t the interest will be compounded.
5. Use the formula above to calculate the ending amount A.

❸ Katherine has $18,000 in her college fund, invested at 4.80% annual interest compounded monthly. How much will she have in 3½ years?

1. $P = 18,000$
2. $r = .048$
3. $n = 12$
4. $t = 3.5$
5. $A = 18,000\left(1 + \frac{.048}{12}\right)^{12(3.5)} = 18,000(1 + .004)^{42} = \$21,286$

As seen in step 5, getting 4.80% interest per year compounded monthly for 3.5 years means getting 0.40% interest per month for 42 months.

In this case, .048 ÷ 12 is exactly .004, so the calculations are fine. It is important not to round the interest rate (or any value with an exponent).

2-D LOGARITHMIC FUNCTIONS

A logarithm is an exponent: The LOGARITHM with base b of a number x is the exponent that will change b into x.

The inverse of an exponential function is a logarithmic function: $\log_b x = y$ means $x = b^y$.

Since inverses cancel each other, $\log_b b^x = x$, and $b^{\log_b x} = x$.

❶ Simplify the composition of a logarithmic function and an exponential function.

1. If not already done, write the expression so that the exponential function and the logarithmic function have the same bases.
2. Cancel the exponential function with the logarithmic function.

❶ a) $\log_4 4^{3x}$ b) $3^{\log_3 2x}$ c) $\log_2 8^x$

a) $3x$ b) $2x$ c) $\log_2 (2^3)^x = \log_2 2^{3x} = 3x$

❷ Evaluate a simple logarithm by hand.

1. Change the problem from logarithmic form ($\log_b x = y$) to exponential form ($x = b^y$) and determine the exponent that makes the statement true.

❷ a) $\log_4 16$ b) $\log_4 \frac{1}{16}$ c) $\log_4 2$ d) $\log_4 \frac{1}{2}$ e) $\log_4 4$ f) $\log_4 1$ g) $\log_4 0$ h) $\log_4 -16$

a) $4^2 = 16$ b) $4^{-2} = \frac{1}{16}$ c) $4^{½} = 2$ d) $4^{-½} = \frac{1}{2}$ e) $4^1 = 4$ f) $4^0 = 1$ g), h) undefined: 4^x cannot be 0 or negative

The most frequently used bases of logs are 2, 10, and e. $e \approx 2.71828$ is an irrational number of great mathematical importance, especially in calculus.

$\log_{10} x$ is called the COMMON Log of x and is typically written "log x".

$\log_e x$ is called the NATURAL Log of x and is typically written "ln x".

❸ Evaluate a common or natural logarithm with a calculator.

1. Use the [LOG] button for base 10.
   Use the [LN] button for base e.

❸ a) log .01 b) log 2.24 c) ln 25 d) log -100

a) -2 b) 0.35 c) 3.22 d) nonreal answer

The following Properties of Logarithms apply to all positive numbers except $b = 1$.

Property	Rule	Example	Related Property
Product	$\log_b xy = \log_b x + \log_b y$	$\log (1000 \cdot 100) = \log 1000 + \log 100 = 3 + 2 = 5$	$b^x b^y = b^{x+y}$
Quotient	$\log_b \frac{x}{y} = \log_b x - \log_b y$	$\log \frac{100{,}000}{100} = \log 100{,}000 - \log 100 = 5 - 2 = 3$	$\frac{b^x}{b^y} = b^{x-y}$
Power	$\log_b x^y = y \log_b x$	$\log 1000^2 = 2 \log 1000 = 2 \cdot 3 = 6$	
Negative	$\log_b \frac{1}{x} = -\log_b x$	$\log_2 \frac{1}{8} = -\log_2 8 = -3$	$\frac{1}{b^x} = b^{-x}$
Reciprocal	$\log_b x = \frac{1}{\log_x b}$	$\log_8 2 = \frac{1}{\log_2 8} = \frac{1}{3}$	
Change of Base	$\log_b x = \frac{\log_a x}{\log_a b}$	$\log_{16} 32 = \frac{\log_2 32}{\log_2 16} = \frac{5}{4}$	

❹ Simplify a logarithmic expression.

1. Use the above properties.

❹ Simplify the following expressions. State the property of logarithms used for each step.

a) $2 \log_3 x + \log_3 10$
$\log_3 x^2 + \log_3 10$ (power property)
$\log_3 10x^2$ (product property)

b) $\log_3 9^{4x}$
$4x \cdot \log_3 9$ (power property)
$4x \cdot 2 = 8x$ (definition)

c) $\log_3 \left(\frac{1}{9}\right)^{4x}$
$4x \cdot \log_3 \frac{1}{9}$ (power property)
$4x \cdot -\log_3 9$ (negative property)
$4x \cdot -2 = -8x$ (definition)

❺ Simplify a logarithm in which the base and argument are both powers of the same base.

1. Identify a new base such that the original base and the argument are both powers of this base.
2. Apply the change of base property, using this new base.

❺ $\log_{81} 27$

1. 81 and 27 are both powers of 3.
2. $\frac{\log_3 27}{\log_3 81} = \frac{3}{4}$

❻ Evaluate any logarithm.

1. Apply the change of base property, using 10 or e as the new base so that the expression can be entered in a calculator.

❻ $\log_{16} 33$

1. $\frac{\log 33}{\log 16} \approx 1.26$

Equations in which an exponential expression can be isolated can be solved by taking the logarithm on each side, using the same base as the exponential.

❼ Solve a basic exponential equation.

1. Isolate the exponential expression.
2. Take the log on each side, using the same base as is used for the exponential.
3. Apply the change of base property.
4. Continue solving.

❼ $3(7^{2x-1}) - 30 = 360$

1. $7^{2x-1} = 130$
2. $\log_7 7^{2x-1} = \log_7 130$
3. $2x - 1 = \frac{\log 130}{\log 7} \approx 2.50$
4. $x \approx 1.75$

Equations in which each side is exponential can be solved by taking the logarithm on each side and using the power property.

❽ Solve an equation in which one exponential expression is equal to another.

1. Take the logarithm on each side. Any base will work, but there will be fewer steps if you use a base with a calculator button (10 or *e*) or a base from the equation.
2. If using base 10 or *e*, apply the power property on each side.
   If using a base from the equation, apply the power property and then then change of base property on the side that still has a logarithm.
3. Use a calculator to evaluate the log expressions.
4. Distribute.
5. Finish solving.

❽ $6^{5x-9} = 4^{8x}$

1. $\log 6^{5x-9} = \log 4^{8x}$
2. $(5x-9)\log 6 = 8x(\log 4)$
3. $(5x-9)(0.778) \approx 8x(0.602)$
4. $3.890x - 7.002 \approx 4.816x$
5. $x \approx 7.562$

To EXPONENTIATE an expression with base *b* is to make the expression an exponent of *b*. Exponentiation is used to solve logarithmic equations.

❾ Solve a logarithmic equation.

1. If the equation uses more than one base, use the change of base formula to get it to a single base.
2. Use basic algebra and properties of logarithms as needed to get a single log expression only on one side.
3. Exponentiate each side of the equation using the same base as the logarithm.
4. Simplify by canceling the exponential and logarithmic functions.
5. Solve. Disregard any solution that is extraneous due to requiring the log of a nonpositive number.

❾ a) $\log_x 400 = 5$	b) $8\log_9 x - 2\log_3 4x = 5$
1.	$8\left(\frac{\log_3 x}{\log_3 9}\right) - 2\log_3 4x = 5$
2.	$8\left(\frac{\log_3 x}{2}\right) - 2\log_3 4x = 5$
	$2\log_3 x - \log_3 4x = 2.5$
	$\log_3 x^2 - \log_3 4x = 2.5$
	$\log_3 \frac{x}{4} = 2.5$
3. $x^{\log_x 400} = x^5$	$3^{\log_3 x/4} = 3^{2.5}$
4. $400 = x^5$	$\frac{x}{4} = 3^{2.5}$
5. $x = 400^{1/5} \approx 3.31$	$x = 4(3^{2.5}) \approx 62.4$

The HALF-LIFE of a substance is the amount of time it takes for half of it to decay into another substance. In a half-life situation, the decay factor per half-life is $b = ½$, and the exponent is the number of half-lives, which is the total time t divided by how long h the half-life is. $f(t) = a\left(\frac{1}{2}\right)^{t/h}$.

❿ Translate a description of a half-life situation into an equation, and solve it.

1. Identify the starting amount, if known. Label it a.
2. Identify the value of the ending amount, if known. It may be known only relative to the starting amount. Label it $f(t)$.
3. Identify the length of one half-life, if known. Label it h.
4. Identify the amount of time, if known, using the same units as for h. Label it t.
5. Solve the equation $f(t) = a\left(\frac{1}{2}\right)^{t/h}$ for the unknown value. Logarithms will be used if solving for t or h.

❿ Polonium-218 has a half-life of 3.11 minutes. How long will it take 400 grams to decay to 30 grams?

1. $a = 400$
2. $f(t) = 30$
3. $h = 3.11$ (in minutes)
4. t is unknown, and it is measured in minutes.
5. $30 = 400\left(\frac{1}{2}\right)^{t/3.11}$

   $.075 = \left(\frac{1}{2}\right)^{t/3.11}$

   $\log_{1/2} .075 = \log_{1/2}\left(\frac{1}{2}\right)^{t/3.11}$

   $\frac{\log .075}{\log ½} = \frac{t}{3.11}$

   $t = 3.11\left(\frac{\log .075}{\log ½}\right) \approx 11.6$ minutes

Half-life is related to decay rate based on the formula $(1 - r)^h = \frac{1}{2}$.

⓫ Determine half-life based on decay rate or decay rate based on half-life.

1. Use the decay rate r or the half-life h in the formula above.
2. To solve for r, put each side to the power of $\frac{1}{h}$ and continue solving algebraically.

   To solve for h, take the logarithm with base $1 - r$ on each side and use the change of base property.

⓫ Polonium-218 has a half-life of 3.11 minutes. What is its rate of decay?

1. $(1 - r)^{3.11} = \frac{1}{2}$
2. $1 - r = \left(\frac{1}{2}\right)^{1/3.11}$

   $r = 1 - \left(\frac{1}{2}\right)^{1/3.11} \approx 20.0\%$ per minute

⓫ Polonium-218 decays at a rate of 20.0% per minute. What is its half-life?

1. $(1 - 0.200)^h = \frac{1}{2}$
2. $\log_{0.8} 0.8^n = \log_{0.8} \frac{1}{2}$

   $h = \frac{\log ½}{\log 0.8} \approx 3.11$ minutes

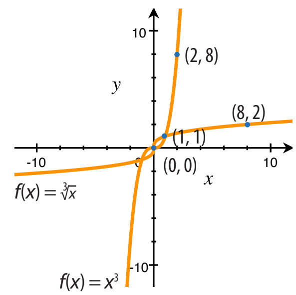
2-E Sketches of Functions

Functions within the same family have similar graphs to each other.

❶ Sketch a power function $f(x) = x^n$.

1. Starting at (0, 0) and passing through (1, 1), and $(2, 2^n)$, sketch a curve that gets steeper as it gets further away from the axes. The higher the power, the steeper it gets.
2. If the power is even, reflect the first quadrant into the second quadrant to make a parabola.
   If the power is odd, rotate the first quadrant into the third quadrant.

❶ Sketch $f(x) = x^3$.



❷ Sketch a root function $f(x) = \sqrt[n]{x}$.

1. Starting at (0, 0) and passing through (1, 1) and $(2^n, 2)$, sketch a curve that gets flatter as it gets further away from the axes. The higher the power, the flatter it gets.
2. If the power is odd, rotate the first quadrant into the third quadrant.

❷ Sketch $f(x) = \sqrt[3]{x}$.

Exponential and logarithmic functions are increasing if the base b is greater than 1 or decreasing if the base b is less than 1. The base cannot be 1, 0, or negative.

An ASYMPTOTE is a line that gets infinitely close to a graph without touching it. $y = b^x$ has an asymptote of $y = 0$, and $y = \log x$ has an asymptote of $x = 0$.

❸ Sketch an increasing exponential function $f(x) = b^x$.

1. Starting at (0, 1) and passing through $(1, b)$ and $(2, b^2)$, sketch a curve that gets steeper as it gets further away from the axes. The higher the base, the steeper it gets.
2. Continue the curve back into the second quadrant, asymptotic to the x-axis.

❹ Sketch a decreasing exponential function $f(x) = b^x$.

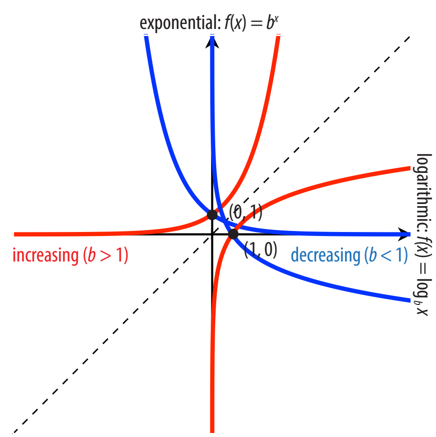
1. Do step 1, above, except make each x value negative.
2. Do step 2, above, except into the first quadrant.

❺ Sketch an increasing logarithmic function $f(x) = \log_b x$.

1. Starting at (1, 0) and passing through $(b, 1)$ and $(b^2, 2)$, sketch a curve that gets flatter as it gets further away from the axes. The higher the base, the flatter it gets.
2. Continue the curve down into the fourth quadrant, asymptotic to the y-axis.

❻ Sketch a decreasing logarithmic function $f(x) = \log_b x$.

1. Do step 1, above, except make each y value negative.
2. Do step 2, above, except into the first quadrant.



2-F Transformations

A TRANSFORMATION takes an original function, called the PRE-IMAGE, and changes its graph's position, size, or direction so that it becomes a new function, called the IMAGE. Four common types of transformations are translations, stretches, reflections, and rotations, the first three of which can be done with basic algebra.

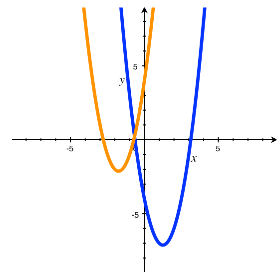
A TRANSLATION changes the *position* of a graph.

$f(x-h)$ is a Horizontal Translation: The graph is moved h units to the right.

$f(x)+k$ is a Vertical Translation: The graph is moved k units upward.

❶ Translate a function h units right and k units up.

1. Subtract h from each x. To translate it left, h will be negative, which will result in adding to x.
   Add k to the expression. To translate it down, k will be negative, which will result in subtracting from the expression.
2. Simplify.



❶ Translate the pre-image $f(x) = 2x^2 - 5x - 4$ three units left and five units up.

1. $f(x+3) + 5 = 2(x+3)^2 - 5(x+3) - 4 + 5$
2. $f(x+3) + 5 = (2x^2 + 12x + 18) - (5x + 15) + 1$
   $f(x+3) + 5 = 2x^2 + 7x + 4$

A STRETCH changes the *size* of the graph by expanding it away from an axis or contracting it toward an axis.

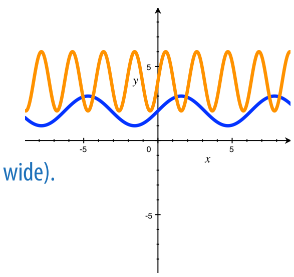
$f\left(\frac{x}{b}\right)$ is a Horizontal Stretch: The graph is expanded away from the y-axis by a factor of b.

$a \cdot f(x)$ is a Vertical Stretch: The graph is expanded away from the x-axis by a factor of a.

If $|a|$ or $|b|$ is less than 1, the graph will contract in that dimension instead of expand.

❷ Stretch a function horizontally by a factor of b and vertically by a factor of a.

1. Divide each x by b and multiply the expression by a.
2. Simplify.



❷ Stretch the pre-image $f(x) = 2 + \sin x$ by a factor of 2 vertically (twice as tall) and by a factor of $\frac{1}{3}$ horizontally (one third as wide).

1. $2 \cdot f\left(\frac{1}{1/3}\right) = 2\left(2 + \sin \frac{1}{1/3}\right)$
2. $2 \cdot f(3x) = 4 + 2 \sin 3x$

A REFLECTION changes the *direction* of a graph by flipping it across a line.

$f(-x)$ is a Horizontal Reflection: The graph is flipped across the y-axis.

$-f(x)$ is a Vertical Reflection: The graph is flipped across the x-axis.

❸ Reflect a function across the y-axis and/or the across the x-axis.

1. To reflect across the y-axis, multiply each x by -1. This is a horizontal reflection.
   To reflect across the x-axis, multiply y (that is, the entire expression, since it is set equal to y) by -1. This is a vertical reflection.
2. Simplify.

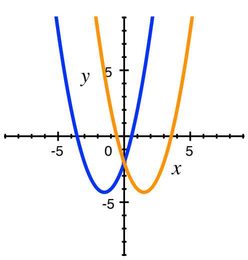
❸ Reflect the pre-image $f(x) = x^2 + 3x - 2$ across the stated axis.

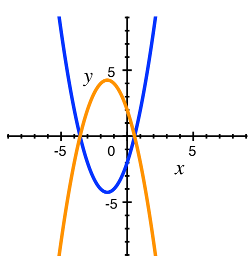
a) the y-axis

1. $f(-x) = (-x)^2 + 3(-x) - 2$
   $f(-x) = x^2 - 3x - 2$

b) the x-axis

2. $-f(x) = -(x^2 + 3x - 2)$
   $-f(x) = -x^2 - 3x + 2$





When different types of transformations in the same dimension are applied, such as a vertical stretch and a vertical reflection, the image will depend on the order in which the transformations are applied.

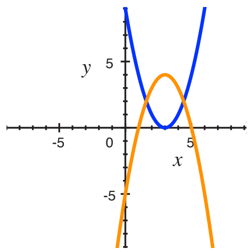
❹ Apply multiple transformations to a function.

1. Do each transformation in the order stated. See ❶, ❷, and ❸.
2. Simplify.

❹ Transform the pre-image $f(x) = x^2 - 6x + 9$ by doing the following.

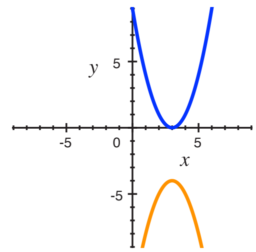
a) Reflect it across the x-axis, and then translate it up four units.

1. $-f(x) = -(x^2 - 6x + 9)$
   $-f(x) + 4 = -(x^2 - 6x + 9) + 4$
2. $-f(x) + 4 = -x^2 + 6x - 5$



b) Translate it up four units, and then reflect it across the x-axis.

$f(x) + 4 = x^2 - 6x + 9 + 4$
$-(f(x) + 4) = -(x^2 - 6x + 9 + 4)$
$-f(x) - 4 = -x^2 + 6x - 13$



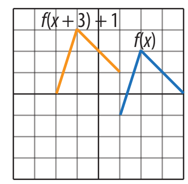
Graphs can be transformed based on the rules above.

❺ Given the graph of $f(x)$, sketch $f(x - h) + k$.

1. Identify h and k.
2. Translate the graph h units to the right and k units up.

❺ Given $f(x)$ in blue, graph $f(x + 3) + 1$.

1. $h = -3$, $k = 1$

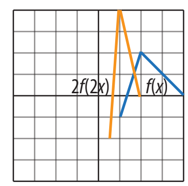


❻ Given the graph of $f(x)$, sketch $a \cdot f\left(\frac{x}{b}\right)$.

1. Identify a and b.
2. Multiply every y coordinate by a: Every point becomes a times as far from the x axis as it was before.
3. Multiply every x coordinate by b: Every point becomes b times as far from the y axis as it was before.

❻ Given $f(x)$ in blue, graph $2f(2x)$.

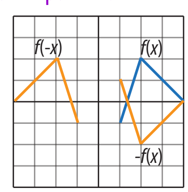
1. $a = 2$, $b = \frac{1}{2}$



❼ Given the graph of $f(x)$, sketch $f(-x)$ or $-f(x)$.

1. For $f(-x)$, reflect the graph horizontally across the y-axis: Every positive x-coordinate becomes negative and every negative x-coordinate becomes positive.
2. For $-f(x)$, reflect the graph vertically across the x-axis: Every positive y-coordinate becomes negative and every negative y-coordinate becomes positive.

❼ Given $f(x)$ in blue, graph $f(-x)$ and $-f(x)$.



The equation of an image can be determined by identifying the transformations that have been applied to the pre-image.

❽ Use the equation of a pre-image to find the equation of a graph.

1. Identify the equation of the pre-image.
2. If the image is vertically reflected, multiply each side of the equation by -1.
3. If the image is horizontally reflected, multiply each x in the equation by -1.
4. If the image is vertically stretched, multiply each side of the equation by the vertical scale factor a.
5. If the image is horizontally stretched, divide each x in the equation by the horizontal scale factor b.
6. If the image is vertically translated, add the distance moved up k (after the reflection, if any) to each side of the equation.
7. If the image is horizontally translated subtract the distance moved right h (after the reflection, if any), from each x in the equation.
8. Simplify.
9. If the image has a function name, you can change the left side of the equation to this.

❽ Write the equation for $g(x)$ graphed at right.

1. The pre-image is labeled as $f(x) = x^2 - 6x + 4$.
2. There is a vertical reflection across the x-axis. $-f(x) = -(x^2 - 6x + 4)$
6. There is a vertical translation: After being reflected, it is moved down 3 units. $-f(x) - 3 = -(x^2 - 6x + 4) - 3$
7. There is a horizontal translation: It is moved left 4 units. $-f(x + 4) - 3 = -((x + 4)^2 - 6(x + 4) + 4) - 3$
8. $-f(x + 4) - 3 = -x^2 - 2x + 1$
9. $g(x) = -x^2 - 2x + 1$