CHAPTER FOUR: PROBABILITY

Review November 4 ➳ Test November 18

Many people find probability to be difficult because it involves more reasoning than straight-forward procedures. On the other hand, many like it because it is more directly applicable than other topics. This chapter finishes with the creation of a class casino in which other students can gamble. In order to make money, you will need to have a positive expected value of each play of your game, which is the average you will gain each time someone plays. This chapter starts by explaining counting methods and conditional probabilities, as these are used to calculate expected values. The law of large numbers says that in the long run the actual average will work out to be close to the expected average, so as long as your game is sufficiently attractive and entertaining to get a lot of players, even a small house advantage will earn you money over time. This is the principle that allows real casinos to pay out huge winnings without worry.

4-A Counting Methods — Friday • 10/21

multiplication principle • permutation • combination • choose • sample space

❶ Use a calculator to count combinations.
❷ Find the total number of possible outcomes in a series of events.
❸ Find the size of a sample space by using permutations, if possible.

4-B Probability of a Single Event — Wednesday • 10/26

set • element • cardinality • intersection • union • complement • universal set • empty set • mutually exclusive • given • conditional probability

❶ Read set notation.
❷ Use the size of a sample space to find the probability of an event.
❸ Find the probability of either of two events.
❹ Find probabilities based on given information.
❺ Make a table to calculate conditional probabilities for two events.

4-C Probability of Multiple Events — Monday • 10/31

dependent events • independent events

❶ Identify whether events are dependent or independent.
❷ Calculate the probability of multiple events.
❸ Calculate the probability of an event that can occur in different ways.
❹ Calculate the probability of an event that can occur in different orders.
❺ Calculate the probability of at least or at most x out of n occurrences of an event.

4-D Expected Value — Friday • 11/4

expected value

❶ Calculate the expected value of a casino event.

Casino — Wednesdy • 11/16

4-A Counting Methods

A COMBINATION is a group of items not assigned specific labels or placements within the group. *N* CHOOSE *R*, written $\binom{n}{r}$ or *nCr*, is the number of combinations of *r* items that can be made from a set of *n* items. $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. Note that $\binom{n}{0} = 1$, $\binom{n}{1} = n$, and $\binom{n}{r} = \binom{n}{n-r}$.

❶ Use a calculator to count combinations.

1. Type the value of *n*, which is the total number of items, places, etc. there are to choose from.
2. Push [MATH] and choose `PRB`.
3. Choose `nCr`.
4. Type the value of *r*, which is the number of items, people, etc., that are being grouped together with a specific label.

❶ State the number of combinations there are for each of the following, and list two possible combinations.

a) 3 out of 10 applicants are selected to win a scholarship.
1. $n = 10$ applicants
4. $r = 3$ winners
`10 nCr 3` = 120 different possible combinations, such as Ted, Jill, and Mark or Talia, Ricky, and Mark

b) 4 coins land on heads and 2 land on tails.
$n = 6$ coins
$r = 4$ heads
`6 nCr 4` = 15 different possible combinations, such as HHHHTT or THTTTH.

A SAMPLE SPACE consists of all the possible outcomes of an event, such as "♠, ♥, ♦, ♣" for suit of a card, or "♠♠♦, ♠♦♠, ♦♠♠" for a combination of two spades and a diamond.

The size of a sample space can be found using choose. In the examples above, there are $\binom{4}{1} = 4$ ways to choose one of the four suits, and there are $\binom{3}{2} = 3$ ways to choose which 2 out of 3 cards are spades.

The MULTIPLICATION PRINCIPLE states that the total number of possible outcomes of a series of events is the product of the sizes of the individual sample spaces.

❷ Find the total number of possible outcomes in a series of events.

1. Identify the size of the sample space of each individual event, using $\binom{n}{r}$ as needed.
2. Multiple these sizes together.

❷ State the number of possible outcomes of the following.

a) Choose 3 representatives out of 9 seniors and 2 representatives out of 8 juniors.
$\binom{9}{3}\binom{8}{2} = 84 \cdot 28 = 2352$

b) Identify the 1st place, 2nd place, 3rd place, and 4th place finisher out of 25 racers.
$\binom{25}{1}\binom{24}{1}\binom{23}{1}\binom{22}{1} = 25 \cdot 24 \cdot 23 \cdot 22 = 303600$

The outcome of a series of events that each involve choosing one item from those remaining, as in example ❷b above, is called a PERMUTATION. This is the same as a combination, except that each item in the group is assigned a specific label or placement within the group. In other words, a combination is a selection of *r* items from a group of *n* items, and a permutation is a selection of *r* items, *chosen one at a time*, from a group of *n* items.

Permutations are never required, but can be used as an easier way to indicate mutliple combinations in which one of the remaining elements is chosen each time. For example, $\binom{9}{1}\binom{8}{1}\binom{7}{1}$ can be written as ${}_9P_3$.

The number of permutations of *r* items that can be made from a set of *n* items is $nPr = \frac{n!}{(n-r)!}$, which can be calculated using `nPr` on the calculator.

❸ Find the size of a sample space by using permutations, if possible.

1. Express the number of possible outcomes as a series of combinations.
2. If each combination involves removing a single item, that is, $\binom{n}{1}\binom{n-1}{1}\binom{n-2}{1}\binom{n-3}{1}\ldots$, the expression can be rewritten as nPr, where r is the total number of items being chosen. Otherwise, permutations do not apply.

❸ Use combinations to express the size of the sample space for each of the following. Then rewrite the solution using permutations if possible, or explain why not.

a) Emma chooses her 3 favorite months.
$\binom{12}{3}$ cannot be written as a permutation, because she is choosing a group of three equal items, not three seperate, distinguishable items.

b) Emma chooses her favorite, second favorite, and third favorite month.
$\binom{12}{1}\binom{11}{1}\binom{10}{1}$ can be written ${}_{12}P_3$.

c) Use a 6-sided die to choose a color for each of three teams.
$\binom{6}{1}\binom{6}{1}\binom{6}{1}$ cannot be written as a permutation, because the colors are not being removed as they are chosen.

d) Put 6 colors in a hat and draw three of them to choose a different color for each of three teams.
$\binom{6}{1}\binom{5}{1}\binom{4}{1}$ can be written ${}_6P_3$.

4-B Probability of a Single Event

A SET is a collection of items, called ELEMENTS. $x \in A$ means x is an element of set A.

The CARDINALITY of a Set A, $|A|$, is the number of elements it contains.

The INTERSECTION of Two Sets A and B, $A \cap B$, is the set of elements that are in both A and B.

The UNION of Two Sets A and B, $A \cup B$, is the set of elements that are in either A or B (or both).

The COMPLEMENT of a Set A, A', is the set of elements not in A.

The UNIVERSAL Set, U, is the set of all elements in the sample space.

The EMPTY Set, $\varnothing$, contains no elements.

❶ Read set notation.

1. Use the definitions above. In general, $\cap$ means *and*, $\cup$ means *or*, ’ means *not*, and | | means *number of*.

❶ State the following in words, given A is the set of aces and B is the set of black cards.

a) A	the aces
b) U	all of the cards
c) $\varnothing$	none of the cards
d) $\lvert A\rvert$	the number of aces
e) $\lvert U\rvert$	the number of cards
f) $A \cup B$	the cards that are an ace or black
g) $A \cap B$	the cards that are an ace and black
h) $\lvert A \cap B\rvert$	the number of black aces
i) A'	the cards that are not aces
j) $(A \cap B)'$	the cards that are not black aces
k) $(A \cup B)'$	the cards that are not an ace or black

In probability, a set is used to represent an event, with the elements in the set being the event’s possible outcomes.

The probability of event A, $P(A)$, is the probability of an outcome in A occuring. $P(A) = \frac{|A|}{|U|}$.

Because $|A|$ and $|U|$ have specific meanings, reducing a probability or converting it to a decimal or a percent causes information to be lost. Therefore, unless directed otherwise, use only fractions for probabilities in this class, and do not reduce probabilities not equal to 0 or 1.

❷ Use the size of a sample space to find the probability of an event.

1. Identify the denominator $|U|$, the size of the sample space. Use combinations if needed (see ❷).
2. Identify the numerator $|A|$, which is the number of possible outcomes for event A within the sample space. Use combinations if needed (see ❷).

❷ Ryan draws two cards. Find the probability that …

a) the first card is a ace

1. There are $|U| = 52$ possible cards.
2. There are $|A| = 4$ possible aces.
3. $P(A) = \frac{4}{52}$

b) both cards are aces

There are $|U| = \binom{52}{2} = 1326$ possible combinations of 2 of the 52 cards.

There are $|A| = \binom{4}{2} = 6$ possible combinations of 2 of the 4 aces.

$P(A) = \frac{6}{1326}$

Two events are said to be MUTUALLY EXCLUSIVE, or Disjoint, if the occurance of one eliminates the possibility of the other, such as clubs and hearts on a single card. If A and B are mutually exclusive, the probability that one of them will happen is $P(A \cup B) = P(A) + P(B)$.

The above formula does not work for events that are not mutually exclusive, because some outcomes would be counted more than once. To account for this, the overlap between the events can be subtracted: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

❸ Find the probability of either of two events.

1. Add the probabilities of the two events.
2. Subtract the probability of the two events happening simultaneously, since this has been double-counted.

❸ Find the probability of a card being as stated.

a) red or an ace

1. $\frac{26}{52} + \frac{4}{52} = \frac{30}{52}$
2. $\frac{30}{52} - \frac{2}{52} = \frac{28}{52}$

b) a 9 or an ace

$\frac{4}{52} + \frac{4}{52} = \frac{8}{52}$

$\frac{8}{52} - 0 = \frac{8}{52}$

Probability is based on what is known, not on what has happened. GIVEN means *known* (or *knowing*). Given information changes the universal set. For example, the probability of a card being hearts, given it is red, is $\frac{13}{26}$, because U now only includes the 26 red cards.

CONDITIONAL Probabilities incorporate given information that would not necessarily be assumed, such as *the card is red*. The probability of event A after knowledge of event B has been taken into account is the probability of A given B: $P(A \mid B) = \frac{|A \cap B|}{|B|}$.

❹ Find probabilities based on given information.

1. Do not assume any information, even for events that have already occurred.
2. Take into account all known (given) information, even for events that have not yet occurred.

❹ Find the following probabilities for Ryan's two cards.

a) The second card is an ace.

1. The first card remains unknown. Do no assume it is or is not an ace. There are still $|U| = 52$ possible outcomes, $|A| = 4$ of which are aces, so $P(A) = \frac{4}{52}$.

b) The second card is an ace, given the first card is an ace.

2. The first card is known, so use that information. There are only $|U| = 51$ remaining possible outcomes for the second card, $|A| = 3$ of which are aces, so $P(A) = \frac{3}{51}$.

c) The first card is an ace, given the second card will be an ace.

2. The second card is known even though it hasn't been flipped yet, so use that information. There are only $|U| = 51$ remaining possible outcomes for the first card, $|A| = 3$ of which are aces, so $P(A) = \frac{3}{51}$.

❺ Make a table to calculate conditional probabilities for two events.

1. Make a row for A and a row for A'. Label them in context.
2. Make a column for B and a column for B'. Label them in context.
3. Multiply to find $P(A \cap B)$, $P(A' \cap B)$, $P(A \cap B')$, and $P(A' \cap B')$, and put these values in the table.
4. Add to find $P(A)$, $P(A')$, $P(B)$, and $P(B')$, and put these values in the table. Verify that $P(A) + P(A')$ 100% and $P(B) + P(B') = 100\%$.
5. Use the formula for conditional probability.

❺ 10% percent of the population has a certain disease. A test for this disease gives a positive result for 70% of people who have the disease and for 20% of people who do not. Show this information in a table, and use it to calculate the probabilities below.

	B: tests positive	B': tests negative	Total
A: has disease	$P(A \cap B) = 10\% \cdot 70\% = 7\%$	$P(A \cap B') = 10\% \cdot 30\% = 3\%$	$P(A) = 10\%$
A': does not have disease	$P(A' \cap B) = 90\% \cdot 20\% = 18\%$	$P(A' \cap B') = 90\% \cdot 80\% = 72\%$	$P(A') = 90\%$
Total	$P(B) = 25\%$	$P(B') = 75\%$	100%

a) a person tests positive: $P(B) = 25\%$

b) a person has the disease and tests positive: $P(A \cap B) = 7\%$

c) a person has the disease, given he or she tests positive: $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{7}{25} = 28\%$

4-C Probability of Multiple Events

The probabilities of DEPENDENT Events are conditional: They are influenced by each others' outcomes.
The probabilities of INDEPENDENT Events do not change.

❶ Identify whether events are dependent or independent.

1. If the probability of events are the same on every trial, they are independent.
2. If the outcome of an event changes the sample space for other events, they are dependent.

❶ Savannah rolls three dice, checking for a 6 each time, and she draws two cards, checking for an ace each time.

1. The die rolls are independent: Each die has a $\frac{1}{6}$ chance of rolling a 6, regardless of the previous rolls.
2. The card draws are dependent: After she checks the first card, there are only 51 remaining possibilities for the second card. The second card has a $\frac{4}{51}$ chance of being an ace if the first card was not an ace, or a $\frac{3}{51}$ chance of being an ace if the first card was an ace.

The probability of both A and B occuring is the product of their probabilities: $P(A \cap B) = P(A) \bullet P(B)$. If the events are dependent, the probability of one of them must be adjusted to its conditional probability based on the other event: $P(A \cap B) = P(A) \bullet P(B \mid A)$, or equivalently, $P(B) \bullet P(A \mid B)$.

❷ Calculate the probability of multiple events.

1. List what has to occur. Ignore events that do not matter.
2. Identify the conditional probability of each event, based on each of the previous events occuring.
3. Multiply all the probabilities together.

❷ Cody draws five cards. Calculate the probability that the first two cards are aces and the fourth card is not an ace.

1. ace, ace, not ace
   (Note that the third and fifth card are irrelevant to this problem.)
2. $P(\text{first card is an ace}) = \frac{4}{52}$
   $P(\text{second card is an ace, given first card is an ace}) = \frac{3}{51}$
   $P(\text{fourth card is not an ace, given first two cards are aces}) = \frac{48}{50}$
3. $\frac{4}{52} \bullet \frac{3}{51} \bullet \frac{48}{50} = \frac{576}{132600}$

Some events can take place in different ways. To calculate the total probability, the probability of each of the possible ways is added together.

❸ Calculate the probability of an event that can occur in different ways.

1. Identify each different way it can occur.
2. Calculate the probability of each way (see ❷, and ❹ if needed).
3. Add these probabilities together.

❸ Sarah grabs 3 random pens from a drawer with 6 black pens, 4 red pens, and 1 purple pen. What is the probability that they are all the same color?

1. It could be black, black, black, or it could be red, red, red.
2. $P(\text{all black}) = \frac{6}{11} \bullet \frac{5}{10} \bullet \frac{4}{9} = \frac{120}{990}$
   $P(\text{all red}) = \frac{4}{11} \bullet \frac{3}{10} \bullet \frac{2}{9} = \frac{24}{990}$
3. $P(\text{all black or all red}) = \frac{120}{990} + \frac{24}{990} = \frac{144}{990}$

When the different ways in which an event can happen are simply different orders of the same thing, each way will have the same probability. Therefore, instead of separately identifying each order and calculating its probability, the probability can be calculated once and multiplied by the number of possible orders.

❹ Calculate the probability of an event that can occur in different orders.

1. Identify one possible order.
2. Calculate the probability of this order.
3. Count or calculate the number of possible orders, using $\binom{n}{r}$ as needed.
4. Multiply the number of possible orders by the probability of each order.

❹ What is the probability that Sarah's 3 pens, above, are all different colors?

1. One possible order is black, red, purple.
2. $P(\text{black, red, purple}) = \frac{6}{11} \bullet \frac{4}{10} \bullet \frac{1}{9} = \frac{24}{990}$
3. There are three spots to choose from for the black pen, two spots left for the red pen, and one remaining for the purple pen, for a total of $\binom{3}{1}\binom{2}{1}\binom{1}{1} = 3 \bullet 2 \bullet 1 = 6$ (that is, ${}_3P_3$) possible orders.
4. $P(\text{all different colors}) = 6 \bullet \frac{24}{990} = \frac{144}{990}$

An event must either happen or not happen. Therefore, these two possibilities are complements of each other, making the sum of their probabilities 100%: $P(A) + P(A') = 1$.

Often it is simpler to calculate $P(A')$ than $P(A)$, especially if when A can happen in more ways than A' can. In this case, $P(A)$ can be found by subtracting its complement from 1: $P(A) = 1 - P(A')$. For example, the probability of getting at least one tails out of four coin flips could be calculated by simply $1 - P(0)$ rather than by $P(1) + P(2) + P(3) + P(4)$.

❺ Calculate the probability of at least or at most x out of n occurances of an event.

1. Count how many ways the event can occur and how many ways its complement can occur to identify whether it will be simpler to calculate the probability of the outcome as stated or of its complement.
2. Calculate the sum of the probabilities identified in step 1 (see ❸).
3. If you calculated the complement, subtract its probability from 1.

❺ Find the probability that out of five 6-sided dice, at least two will roll a 6.

1. The outcome as stated involves the sum of four different probabilities: $P(2) + P(3) + P(4) + P(5)$. The complement is only two: $P(0) + P(1)$.
2. $P(0) = \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} = \frac{3125}{7776}$
   $P(1) = \binom{5}{1} \frac{1}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} = \frac{3125}{7776}$
   $P(0) + P(1) = \frac{3125}{7776} + \frac{3125}{7776} = \frac{6250}{7776}$
3. $P(\geq 2) = \frac{7776}{7776} - \frac{6250}{7776} = \frac{1526}{7776}$

4-D Expected Value

The EXPECTED VALUE of an event is $\mu = \sum xP(x)$, which is the theoretical average result.

When betting against a casino ("the house"), the bettor's expected value is the average amount expected to be gained overall on each bet. It can be found by subtracting the price of the game from the value calculated above. This is the same as the average amount expected to be lost overall by the house on each bet. It is commonly negative, meaning the bettor is expected to lose money and the house is expected to gain money.

❶ Calculate the expected value of a casino event.

1. Identify the probability of each outcome.
2. Multiply the probability of each outcome by its value.
3. Add these products together.
4. If there is a per-event price, subtract it from the sum.

❶ Neil pays $5 to draw a card. He wins $10 if it is a king, and he wins $100 if it is a black ace.

Event	Probability	Value	Product	
king	$\frac{4}{52}$	$10	$0.77	
black ace	$\frac{2}{52}$	$100	$3.85	
other		$0	$0.00	
			$4.62	He wins an average of $4.62 each game.
			– $5.00	He pays $5.00 to play each game.
			-$0.38	Overall, he loses an average of $0.38 each game.

Note that $0.38 can be written as 38¢ (38 cents) but not as .38¢ (less than one cent).