Chapter Five

Exponential and Logarithmic Functions

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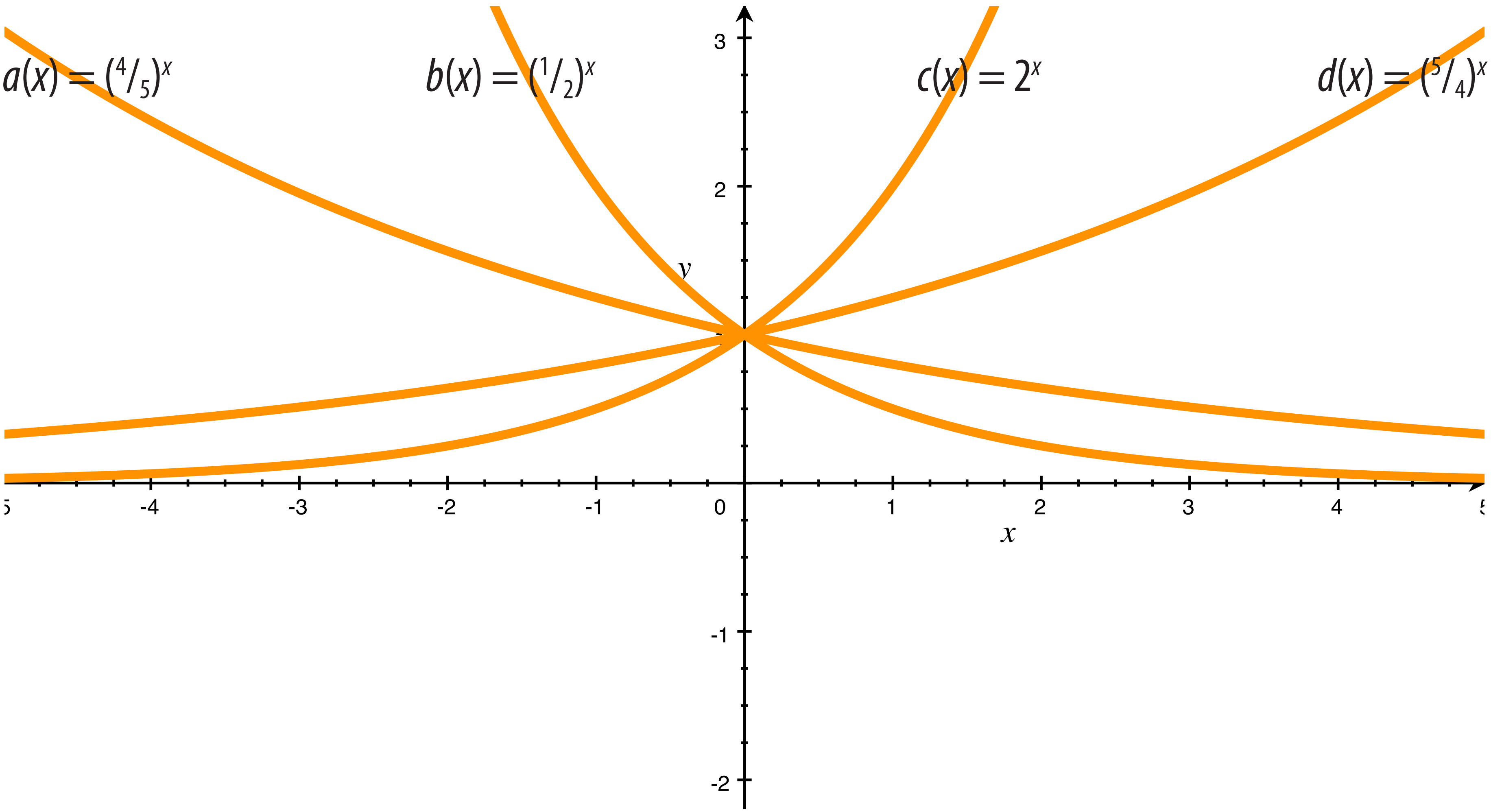
Exponential Functions

In an **exponential function**, the variable is in the exponent: $f(x) = b^x$.

The base b can be any positive number other than 1. The closer b is to 1, the closer the graph is to a horizontal line. The graph increases if b is greater than 1, or decreases if b is less than 1.



Exponential Growth and Decay

A **factor** is a value being multiplied. In $f(x) = b^x$, the base b is a factor.

If b is above 1, it is a **growth factor**, and the amout it is above 1 is the **growth rate**.

If b is below 1, it is a **decay factor**, and the amout it is below 1 is the **decay rate**.

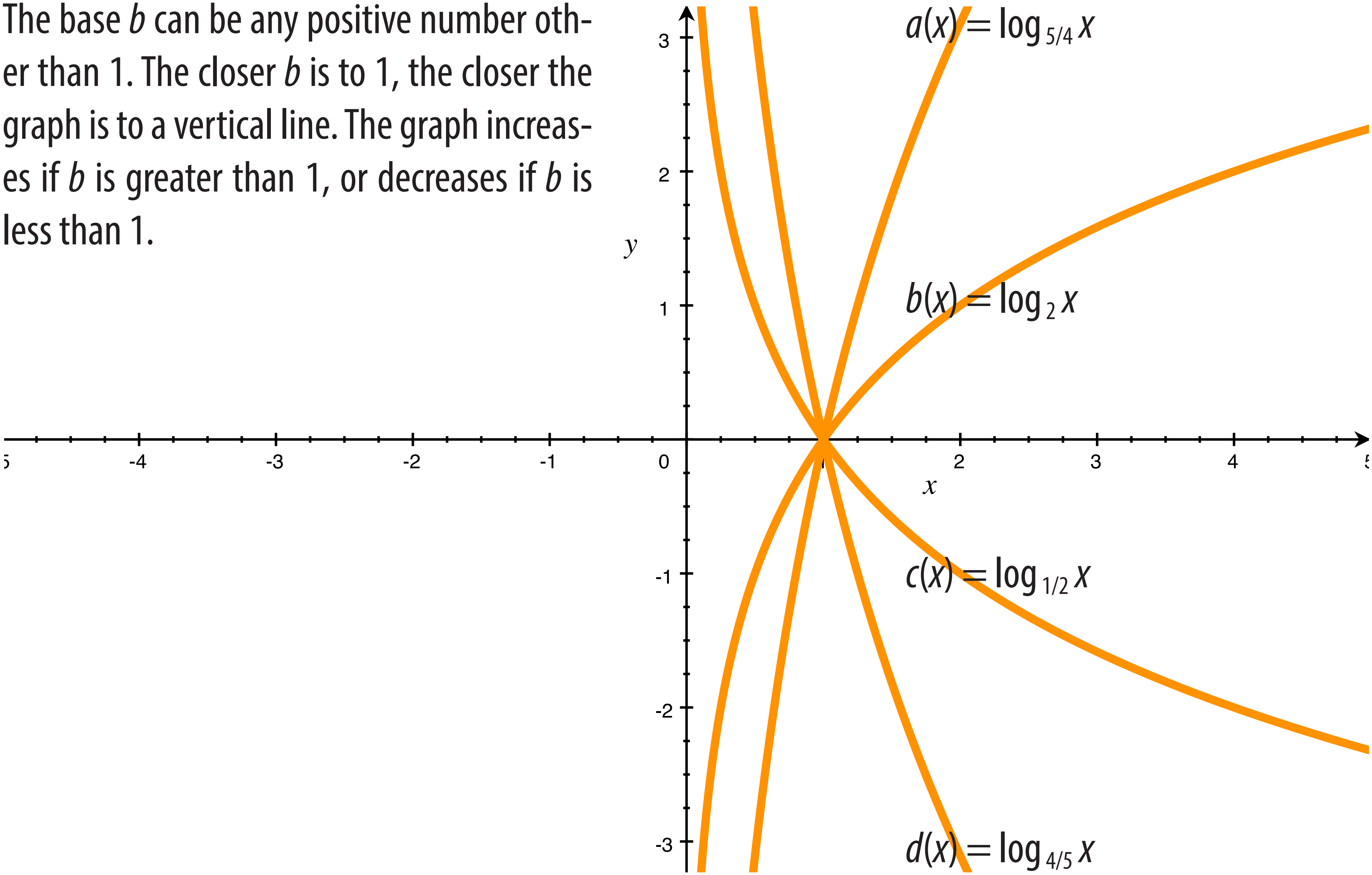
Change, in words	Growth Rate (r) or Decay Rate (r)	Growth Factor ($b = 1 + r$) or Decay Factor ($b = 1 - r$)
8% increase		
8% decrease		
.081% decrease		
increase by half		
increase by half a percent		
75% more		
175% more		
tripling		

A common mistake is to forget to add 1 to growth rates when they are large such as 175%. $f(x) = 1.75^x$ represents only 75% growth. It should be $f(x) = 2.75^x$ to represent 175% growth.

Logarithmic Functions

The inverse of an exponential function is a **logarithmic function**: $f(x) = \log_b x$.

The base b can be any positive number other than 1. The closer b is to 1, the closer the graph is to a vertical line. The graph increases if b is greater than 1, or decreases if b is less than 1.



Logarithms

A logarithm is an exponent. Specifically, the **logarithm** with base *b* of a number *x* is the exponent needed to change *b* into *x*. A simple logarithmic equation can be rewritten as an exponential equation by expressing the value of the logarithm as an exponent: $\log_b m = n \Leftrightarrow b^n = m$. Some examples are shown at below.

Logarithmic form	Exponential form	Value
$\log_4 16 = x$		
$\log_{16} 4 = x$		
$\log_4 {}^1/_{16} = x$		
$\log_4 -16 = x$		

The most common logarithm bases are 2, *e*, and 10. The irrational number $e \approx 2.718$ is used extensively in calculus.

Name	Base	Notation
binary log		
natural log		
common log		

Properties of Logarithms

Property	Rule	Example
Product	$\log_b x + \log_b y = \log_b xy$	$\log_6 4 + \log_6 9 =$
Quotient	$\log_b x - \log_b y = \log_b \frac{x}{y}$	$\log_2 88 - \log_2 11 =$
Power	$\log_b x^y = y \log_b x$	$\log_2 8^5 =$
Negative	$\log_b \frac{1}{x} = -\log_b x$	$\log_2 \frac{1}{16} =$
Reciprocal	$\log_b x = \frac{1}{\log_x b}$	$\log_{16} 2 =$
Change of Base	$\log_b x = \frac{\log_a x}{\log_a b}$	$\log_8 16 =$

Solving simple exponential and logarithmic equations

Like all equations, exponential and logarthmic equations can be solved by applying their inverse to each side. **Exponentiation** is making an expression an exponent of a base.

Type of equation	Approach	Example	Inverse applied	Solution
Exponential	Take the log of each side, using the base used for the exponential expression.	$2^x = 64$		
Logarithmic	Exponentiate each side, using the base used for the logarithmic expression.	$\log_2 x = 6$		

Solving complicated exponential equations

An exponential expression in an equation should be isolated before its inverse is applied. Then the equation can be solved by taking the log of each side, simplifying, and algebraic manipulation.

$10(2^{5x-6}) + 200 = 5000$	Step	Comment
$2^{5x-6} = 480$		This isolates the exponential expression.
$\log_2 2^{5x-6} = \log_2 480$		Use the same base (2) as the exponential.
$5x - 6 = \log_2 480$		Log with base 2 cancels an exponential with base 2.
$5x - 6 = \frac{\log 480}{\log 2}$		Change the base to 10.
$5x - 6 \approx 8.91$		Base 10 logs can be evaluated directly with a calculator.
$x \approx 2.98$		Finish solving with basic algebra.

Solving equations with two exponential expressions

If both sides of an equation are exponential, the common or natural log can be taken on each side, allowing for simplification using the power property.

$2^{5x+9} = 20^{2x}$	Step
$\log 2^{5x+9} = \log 20^{2x}$	
$(5x+9)\log 2 = 2x \log 20$	
$(5x+9)(.30) \approx 2x(1.30)$	
$1.5x + 2.7 \approx 2.6x$	
$x \approx 2.45$	

Common mistakes with exponentials and logarithms

Scenario	Common Mistake	Issue
Solve $3^x = 9$	$\log_3 3^x = \log_3 9$	
Simplify $\log x - \log 3y$	$\frac{\log x}{\log 3y}$	
Solve $8(½)^x = 6$	$\log_4 4^x = \log_4 6$	
Solve $x^{200} = 18$	$x = 18^{1/200} = 1.01$	
Simplify $3 \log 4x$	$\log 4x^3$	
Simplify $\log 64x^3$	$3 \log 64x$	

Exponential situations

The equation $f(x) = ab^x$ can be used to model exponential situations. Keep in mind that $b = 1 + r$ for growth and $b = 1 - r$ for decay.

In the example below, 500 grams (a) of a substance that decays at a rate of 8% per day (r) will decay down to 41 grams ($f(x)$) in 30 days (x).

Variable	Meaning	How to solve for	Example
a	starting amount (amount at time 0)	Divide each side by b^x.	$41 = a(.92)^{30}$ $a =$
b	growth or decay factor	Divide each side by a, and then take the power of $\frac{1}{x}$ on each side.	$41 = 500(b)^{30}$ $b =$
x	time	Divide each side by a, and then take the log on each side using base b.	$41 = 500(.92)^x$ $x =$
$f(x)$	ending amount (amount at time x)	Evaluate ab^x.	$f(30) = 500(.92)^{30}$ $f(30) =$

Solving complicated logarithmic equations

Like exponential expressions, a logarithmic expression in an equation should be isolated before its inverse is applied.

If there is more than one logarithmic expression, properties of logarithms can be used to combine them into a single logarithmic expression.

The change of base property can be used to make the bases the same if needed.

$18 \log_8 3x - \log_2 6x = 16$	Step
$18 \frac{\log_2 3x}{\log_2 8} - \log_2 6x = 16$	
$18 \frac{\log_2 3x}{3} - \log_2 6x = 16$	
$6 \log_2 3x - \log_2 6x = 16$	
$\log_2 (3x)^6 - \log_2 6x = 16$	
$\log_2 \frac{(3x)^6}{6x} = 16$	
$\log_2 121.5x^5 = 16$	
$2^{\log_2 121.5x^5} = 2^{16}$	
$121.5x^5 = 65{,}536$	
$x \approx 3.52$	

Chapter Six

Probability

Counting Methods

Set Notation and Venn Diagrams

Probability of a Single Event

Probability of Specific Multiple Events

Probability of General Multiple Events

Probability Distributions

Combinations

A **combination** is a group of selected items.

The number of possible combinations of *r* items in a group of *n* items is ***n* choose *r***, written $\binom{n}{r}$ or $_nC_r$.

Group of *n* elements	Chosen group of *r* elements	Combination Example	Number of possible combinations
7 days of the week	1 day	Thursday	
12 months of the year	4 months	June, September, October, December	

The **counting principle** states that if two independent events have *a* and *b* possible outcomes, respectively, then there are a total of ***ab*** possible outcomes for the two events. This can be expanded to *abc* possible outcomes for three events, etc.

Events	Outcome Example	Number of possible outcomes
1 day of the week and 1 month of the year	Thursday; October	
1 day of the week and 4 months of the year	Thursday; June, September, October, December	

Choosing one element at a time

Events	Outcome Example	Number of possible outcomes
Roll 3 six-sided dice	6; 3; 5	
Choose 3 favorite colors, in order, out of 8 colors	black; red; orange	
Choose a background color, a border color, and a text color, out of 8 colors	red; black; white	

For independent events (values can be repeated) like the dice example, the calculation can be simplified to the exponential n^r, in this case 6^3.

For dependent events (values cannot be repeated) like the colors examples, the calculation can be simplified to the permutation ${}_nP_r$, in this case ${}_8P_3$. A **permutation** is a combination in which each item selected is assigned a specific value, such as *first*, *second*, and *third*, or *background*, *border*, and *text*.

Set Notation

A **set** is a combination.

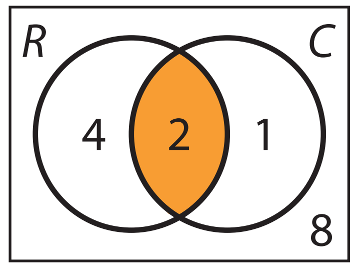
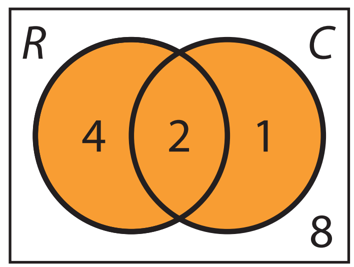
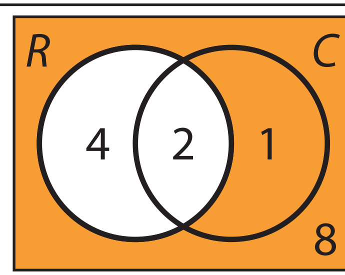
The examples below use the sets $A = \{\text{Saturday, Sunday}\}$ and $B = \{\text{Sunday, Tuesday, Thursday}\}$.

Term	Definition	Notation and Example
Element of *A*	item in *A*	Saturday $\in A$
Cardinality of *A*	number of elements in *A*	$\lvert A \rvert =$
Intersection of *A* and *B*	set of elements in both *A* and *B*	$A \cap B =$
Union of *A* and *B*	set of elements in either *A* or *B*	$A \cup B =$
Complement of *A*	set of elements not in *A*	$A' =$
Universal Set	set of all elements in the given context	$U =$
Empty Set	set containing no elements	$\emptyset =$

Venn Diagrams

A **Venn diagram** is used to represent the relationship between two or three sets, each of which is represented by a circle.

The examples below represent a bag of 6 red marbles (set *R*) and 9 yellow marbles, including 2 red marbles and 1 yellow marble that are cracked (set *C*).

Set notation	Meaning	Venn Diagram	Number of marbles
$R \cap C$	red and cracked		
$R \cup C$	red or cracked		
R'	not red		

The probability of an event

The **sample space** of an event is the set of all possible outcomes.

The probability of an event A, written $P(A)$, can be defined as the number of outcomes satsifying A divided by the total number of outcomes in the sample space: $P(A) = \frac{|A|}{|U|}$. For this definition to apply, the outcomes must all be equally likely.

Event A	Outcomes satisfying A	Sample space	Probability of A
Roll higher than 2 on a 6-sided die	3, 4, 5, 6	1, 2, 3, 4, 5, 6	
Out of three coin flips, two are heads and one is tails.	HHT, HTH, THH	HHH, HHT, HTH, THH, HTT, THT, TTH, TTT	

It is not incorrect to reduce probabilities or convert them to decimals or percents, but doing so removes information about the event. Do not do so in this course, except for fractions equaling 0 or 1.

Given information

Probability is based on the information known, regardless of what has happened. **Conditional** probability takes into account **given** (known) conditions.

Event	Probability	Given condition	Conditional probability
A card is hearts.		The card is red.	
The second card drawn is hearts.		The first card was hearts.	
The third card drawn is hearts.		The next 12 cards are all hearts.	

Probabilities of multiple events

For probability problems involving multiple events, the individual probabilities can be multiplied together. Keep in mind that, in some cases, the individual probabilities change based on the events already accounted for, such as fewer cards being left in a deck as more cards are drawn. Such events are called **dependent events**.

Events	Type	Probability	Comment
Roll 3 dice; all land on 6.			Each die roll is unaffected by the others.
Draw 3 cards; all are aces.			Once you know an ace has been drawn, there are only 3 aces possible to draw out of the remaining 51 cards.
Draw 3 cards, putting them back each time; all are aces.			There are always 4 aces available. You could draw the same ace each time.
Draw 3 cards; they are hearts, clubs, king, in that order.			Even though you know you drew a heart and a club, you have no information about whether or not you drew a king. All 52 cards are still equally likely to be a king.

Different arrangements of multiple events

Some events can happen in different orders or ways. For example, rolling a 2 and a 5 on two dice could be (2, 5) or (5, 2). To find the probability of such an event, every possible order or way must be included.

In many cases, each possibility has the same probability, so can be calculated just once and then multiplied by the number of possibilities there are. This number can be found using choose.

The examples below refer to pulling three marbles out of a jar of 5 orange, 3 red, and 1 black marble.

Event	Possibilities	Probability
All three marbles are the same color.		
Two marbles are red and one is green.		
All three marbles are different colors.		

Using the complement

In many cases, the probability of an event's complement is easier to calculate than the probability of the event itself. Since the complement is everything the original event is not, these two probabilities must add up to 100%: $P(A) = 1 - P(A')$. For example, if there is an 80% chance it will rain today, there is a 20% chance it will not rain today.

The examples below refer to rolling four 6-sided dice.

Event A	Complement A'	$P(A')$	$P(A)$
Roll at least one 6.			
Roll at least two 6's.			

Binomial Probabilities

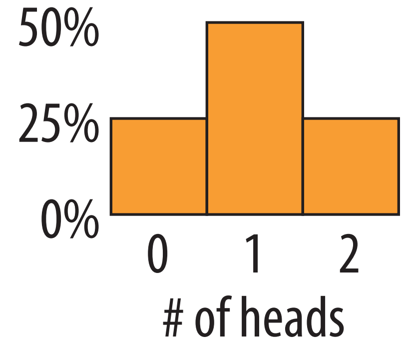
A **binomial experiment** is a scenario in which a specific independent event is attempted multiple times so see how many successes there are.

Value	Meaning	Example: 3 correct predictions in ten 6-sided die rolls
n	number of trials	
r	number of successes	
p	probability of success on each individual trial	
q	probability of failure on each individual trial ($q = p'$)	
p^r	probability of r successes out of r trials	
q^{n-r}	probability of $n - r$ failures out of n trials	
$\binom{n}{r}$	number of possible orders of r successes out of n total trials	
$\binom{n}{r}p^r q^{n-r}$	probability of exactly r successes (and $n - r$ failures) out of n trials	

Probability Distributions

A **probability distribution** states each possible outcome or range of outcomes of an event and how likely it is.

A probability distribution can be displayed in many ways, such as a sentence, table, or graph. When the variable is numerical, a histogram is commonly used.

Flip two coins.	Sentence	Table	Graph
Probability distribution for number of heads	There is a 25% chance of getting either 0 or 2 heads, and a 50% chance of getting exactly 1 head.	Probability of exactly r heads out of 2 coin flips: r | $P(r)$ 0 | 25% 1 | 50% 2 | 25%	Probability of exactly r heads out of 2 coin flips 

Expected Value

The **expected value** μ of a probability distribution is the expected average of infinitely many random values of the distribution. For example, if you win \$10 for tails and \$30 for heads, your expected value is \$20.

Expected value can be calculated by multiplying each possible value of the distribution by its probability and adding these products. The example below calculates the expected number of 4's out of 2 rolls on a 4-sided die.

Event x	Value	Probability $P(x)$	Product $xP(x)$
no 4's	0	$\binom{2}{0}\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = \frac{9}{16}$	
one 4	1	$\binom{2}{1}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = \frac{6}{16}$	
two 4's	2	$\binom{2}{2}\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{16}$	
Total:			

For binomial distributions, expected value can be calculated simply as $\mu = np$. In the example above, $\mu = 2\left(\frac{1}{4}\right) = 0.5$.

Chapter Seven

Linear Correlation

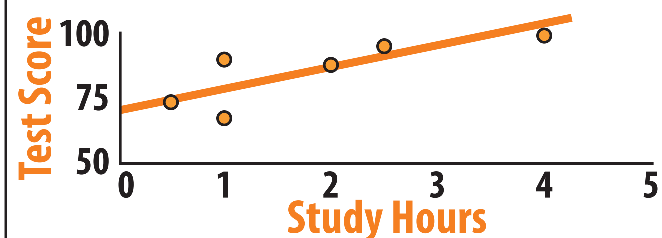
The Line of Best Fit

Statistically Significant Correlations

Causal Relationships

Linear Regression

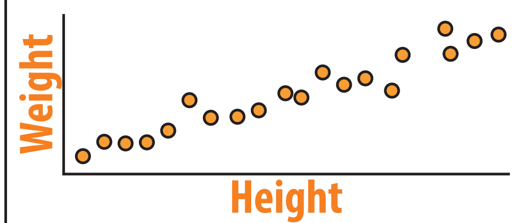
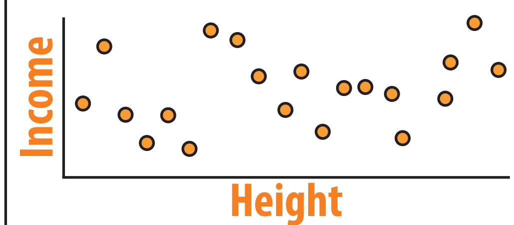
Linear regression is finding the line that best fits a data set. It is used to predict future data values.

Concept	Definition	Example
Paired Data	data values such that each data point has an *x* value and a *y* value	Study hours: 0.5 1.0 1.0 2.0 2.5 4.0 Test score: 74 68 90 88 95 99
Line of Best Fit	the line that best fits the paired data (which must be numerical) by having the smallest sum of squared residuals (see notes for directions on how to calculate it)	
Interpolation	predicting a *y* value for an *x* value between the lowest and highest *x* value in the data set	The predicted test score of a student who studied three hours is $\hat{y} =$
Extrapolation	predicting a *y* value for an *x* value below the lowest *x* value or above the highest *x* value in the data set	The predicted test score of a student who studied ten hours is $\hat{y} =$ *
Residual	the amount a *y* value is higher than would have been predicted by the line of best fit	The residual for the student who studied for two hours is

* Extrapolation commonly leads to unrealistic predictions and should be used with caution.

The Correlation Coefficient

The **correlation coefficient** ***r*** is a value between -1 and 1 that summarizes the strength and direction of the relationship between the two variables in the sample.

Value of *r*	Correlation	Meaning	Example
Positive	positive		
Negative	negative		
Close to 1 or -1	strong		
Close to 0	weak		

Samples and Populations

Data are collected from **samples**, but researchers want to know about entire **populations**.

Term	Definition	Example
Population	the group being studied	Americans
Sample	the subset of the population from which data are actually collected	300 students in an introductory college chemistry class

Ideally, sample data fairly represent the overall population so that conclusions about the population can be made from the sample. Such conclusions may be limited, however, by both random and systematic error.

Cause of Error	Type	Definition	Example
Nonrandom Selection	systematic	the sample is not randomly drawn from the population	College students have more motivation, intelligence, and parental support than the average American.
Coincidence	random	the sample is small enough that a few values that do not fit the trend in the population lead to a misleading conclusion	Coincidentally, several people in the sample exercise a lot but have high blood pressure.

P Values

The ***P* value** of a sample is the probability that another random sample of the same size would, coincidentally, show at least as strong of a result in the hypothesized direction. Coincidence are common in small samples, so large *p* values are common for small samples. Some examples are shown below.

Hypothesis	Result	*P* value	Meaning
Coins land on tails more than heads.	9 out of 15 coins land on tails.	$p = .15$	If coins really land on tails exactly half the time, another sample of 15 coins would still have a 15% chance of getting at least 9 tails.
People watch more than 20 hours of TV per week on average.	6 people watched an average of 24 hours of TV per week, with standard deviation 10 hours.	$p = .19$	If people really watch an average of only 20 hours of TV per week, another sample of 6 people would still have a 19% chance of averaging at least 24 hours.
Students with higher homework scores get higher test scores.	The correlation coefficient for the students in last year's class was $r = .69$.	$p < .01$	

Statistical Signifigance

It is impossible to determine from a sample whether or not the result applies to the population as well, as opposed to being a coincidence. However, the lower the *p* value is, the less likely the results are coincidental. If **$p < .05$**, the results are considered **statistically significant**, and the researchers conclude that their hypothesis was correct not only for their sample (which they know) but also for the population overall (which they could be wrong about).

In the example below, researchers collect a sample of children to see if their grades tend to be higher the more they play outdoors, and although they find their hypothesis to be correct in their sample, it is possible that this is a coincidence there really is no correlation among children in general.

***p* value**	**Conclusion**	**Possible Error**
below 5%	Children tend to get higher grades the more they play outdoors.	
above 5%	There's not sufficient evidence to conclude that children tend to get higher grades the more they play outdoors.	

Data Snooping

Sample data can be used to form hypotheses or to test hypotheses, but when the data used to test a hypothesis were the same data used to come up with it, the researcher has gone in a circle and the *p* value will be meaningless. ***P* values only have meaning for hypotheses that were specifically stated prior to knowing the data.** Searching for any possible pattern within a data set is called **data snooping**, and it is a common fallacy among people unfamiliar with research methods to assume that any found pattern is likely to be legitimate rather than coincidental.

Correlation was predicted?	Conclusion	Action
yes	The correlation likely does exist in the population overall.	
no	The correlation exists in the sample, but there is no evidence that it exists in the population overall.	

Correlation and Causation

When a correlation between two variables is found, it is tempting to conclude that correlation is due to one variable affecting the other. However, in many cases, the correlation is partially or entirely due to outside variables affecting the independent and dependent variable simultaneously. A **confounding variable** is one that affects the dependent variable and is correlated with, but not affected by, the independent variable. Because confounding variables can provide alternative explanations for why one variable is correlated with another, **correlation does not imply causation**: Knowing that a correlation exists is not the same as knowing *why* it exists, such as in the examples below.

Correlation	Presumed reason	Confounding variables	Alternative explanation for correlation
years of education & annual salary	More schooling provides more skills, understandings, and opportunities for better jobs.	motivation, intelligence	
exercise & health	Exercising is good for your health.	caring about health	

Causes of a sample correlation

There are four main categories of explanations for why a correlation exists in a sample. Multiple reasons may apply to a single correlation. For example, a professor may find that the closer students choose to sit to the front of the room in his calculus class, the better their grade tends to be.

Category	Definition	Possibility for seating example
Coincidence	The results in the sample coincidentally do not represent the population. Another sample would probably have much different results.	
Causation	The independent variable affects the dependent variable, as assumed.	
Reverse Causation	The dependent variable affects the independent variable.	
Confounding Variables	Outside variables affect the dependent variable in the way that the independent variable was expected to affect the dependent variable.	

Usefulness of sample data

Four main factors determine how well findings in sample data can be applied to the population. For example, a professor may find that the closer students choose to sit to the front of the room in his calculus class, the better their grade tends to be.

Factor	Limitation without it	Possibility in seating example
Appropriate sampling	The population represented by the sample is only a subset of the entire population.	
Strong correlation	The trend is real, but the size of the effect is small.	
Statistical significance	The trend could easily be coincidental in the sample and not true for the population overall.	
Free from counfounding variables	The trend is true in the population overall, but not for the reason believed.	

Affect and Effect

Discussions of causation frequently use forms of the words *affect* and *effect*.

Word	Part of speech	Clarification	Examples
Affect(s)	verb	has a subject, which is usually one of the following: • an independent variable such as *age* • a confounding variable such as *socioeconomic status*	Smoking affects health. Childhood experiences affect adult personality.
Effect(s)	noun	usually preceded by one of the following: • the article *the* or *an* • an adjective such as *significant* or *two* • a possessive such as *religion's* or *its*	Alcohol has multiple effects. The data demonstrate music's effect on concentration.

Chapter Eight

Vectors

Two-Dimensional Vectors

Vector Equations

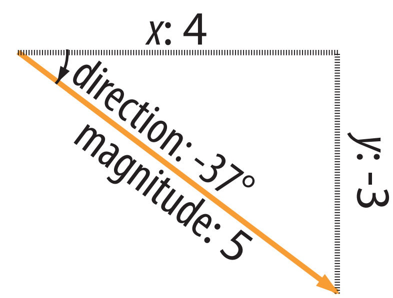
Three-Dimensional Vectors

Unit Vectors

Angles Between Vectors

Two-dimensional vectors

A **vector** is a magnitude and a direction. A vector is symbolized by a bold letter, such as **v**. When handwritten, a vector is symbolized by a letter with a right-facing arrow above it, such as $\vec{v}$.



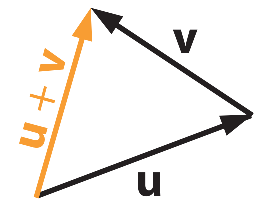
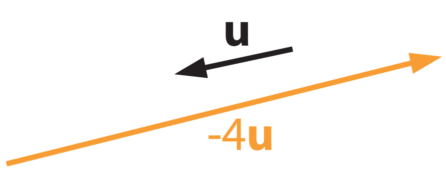
A vector can be represented graphically by an arrow, or numerically by its components. When represented by its components, it is commony written $<x, y>$, $\begin{bmatrix} x \\ y \end{bmatrix}$, or $\begin{pmatrix} x \\ y \end{pmatrix}$.

Concept	Definition	How to find	Example: v = <4, -3>
x-component	total distance the vector extends horizontally	given	
y-component	total distance the vector extends vertically	given	
magnitude	"length" of the vector	$\lVert \mathbf{v} \rVert = \sqrt{x^2 + y^2}$	
direction	angle made with the positive *x*-axis	$\theta = \tan^{-1} \frac{y}{x}$ (add 180° if x is negative)	

Basic vector arithmetic

Vectors can be added to other vectors.

Vectors can be multiplied by a number, called a **scalar**, to change their magnitude. If a scalar is negative, it will reverse the direction of the vector.

Operation	Algebraic	Geometric
Addition	Add the x components together, and add the y components together. $\begin{pmatrix}8\\3\end{pmatrix} + \begin{pmatrix}-6\\4\end{pmatrix} = \begin{pmatrix}2\\7\end{pmatrix}$	Sketch the second vector starting where the first one ends. Connect the start of the first to the end of the second. 
Scalar Multiplication	Multiply each component by the scalar. $-4\begin{pmatrix}8\\3\end{pmatrix} = \begin{pmatrix}-32\\-12\end{pmatrix}$	Make the vector k times as long, where k is the scalar. If k is negative, reverse its direction. 

Vector equations

Although vectors themselves do not have position, a **position vector** shows the position of a point with respect the origin. The position vector $<x, y>$ represents the point (x, y).

With this definition, a line can be determined by a vector equation.

Aspect	Regular	Vector
Equation	$y = mx + b$	$\mathbf{r} = <a, b> + t<x, y>$
Slope	m	$^{y}/_{x}$
Known Point	$(0, b)$	(a, b)
Independent Variable	x	t
Dependent Variable	y	$\mathbf{r}$
Does the line pass through the point (c, d)?	Plug in $x = c$. If this makes if y equal to d, the point is on the line.	Solve for t in the equations $a + tx = c$ and $b + ty = d$. If t is the same in both equations, the point is on the line.
Where do two lines intersect?	Solve for x in the equation $m_1x + b_1 = m_2x + b_2$ and plug it in to find y.	Solve for t_1 or t_2 in the system $\begin{cases} a_1 + t_1x_1 = a_2 + t_2x_2 \\ b_1 + t_1y_1 = b_2 + t_2y_2 \end{cases}$ and plug it in.

Three-dimensional vectors

Adding in a *z* component makes a vector three-dimensional.

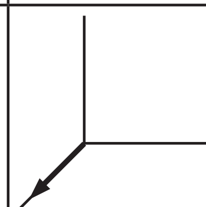
When applicable, methods used for two-dimensional vectors apply to three-dimensional vectors as well, such as in the examples below.

Concept	Two-dimensional example	Three-dimensional example
Vector from a point to a point	The vector from (4, 10) to (5, 7) is $\begin{pmatrix} 5-4 \\ 7-10 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$	The vector from (4, 10, 3) to (5, 7, 9) is
Magnitude	The magnitude of <4, 8> is $\sqrt{4^2+8^2} = \sqrt{80} \approx 8.94$	The magnitude of <4, 8, 3> is
Addition	$2\begin{pmatrix} 4 \\ 8 \end{pmatrix} + 3\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$	$2\begin{pmatrix} 4 \\ 8 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} =$

Unit Vectors

A **unit vector** is a vector with a magnitude of 1.

Three specific unit vectors have their own names.

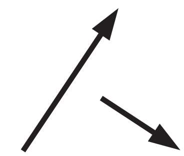
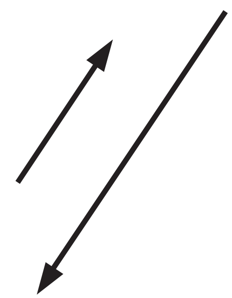
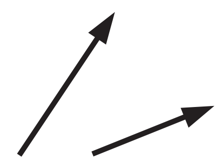
Unit Vector	Component Form	Sketch
i	<1, 0, 0>	
j	<0, 1, 0>	
k	<0, 0, 1>	

Any vector can be written as the sum of multiples of **i**, **j**, and **k**. For example, $<3, -2, 1> = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

A vector divided by its magnitude is a unit vector in the same direction. For example, if $\mathbf{v} = <3, 4>$, then $\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = 5$, and $\frac{\mathbf{v}}{\|\mathbf{v}\|} = <\frac{3}{5}, \frac{4}{5}>$ is a unit vector.

Angles between vectors

The **dot product** of two vectors is the sum of the products of the corresponding components. For example, <3, 10> • <-2, 5> = 3(-2) + 10(5) = -6 + 50 = 44.

Relationship	How to find	Example
Orthogonal	If the dot product is zero, the angle between the vectors is 90°.	**u** = <4, 6> **v** = <3, -2>  **u** • **v** =
Parallel	If one vector is a scalar multiple of the other, the angle between them is 0° if the scalar is positive or 180° if it is negative.	**u** = <4, 6> **v** = <-8, -12> **v** = 
Neither	$\theta = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{\lVert\mathbf{u}\rVert \, \lVert\mathbf{v}\rVert}$	**u** = <4, 6> **v** = <5, 2> θ = 

Chapter Nine

Derivatives

Sequences and Series

Limits

Slopes of Curves

Derivatives

Summation Notation

The $\sum$ symbol is used for summation. Every integer from a starting number (at the bottom) to an ending number (at the top) is plugged into the expression.

The list of resulting values is a **sequence**.

The summation of the resulting values is a **series**, which is what $\sum$ notation represents.

Example	Numbers plugged in	Sequence	Series
$\sum_{x=1}^{4} 10x$			
$\sum_{x=6}^{8} (x^2 + 2x - 10)$			
$\sum_{x=60}^{100} (2x + 1)$			

Two common types of sequences and series are arithmetic and geometric.

Arithmetic sequences have a constant difference *d* from each term to the next.

Geometric sequences have a constant ratio *r* from each term to the next.

Arithmetic and Geometric Sequences

Type	n^{th} term	Explanation	Example
Arithmetic	$A_n = A_1 + (n - 1)d$	Each term is d more than the term before it. Starting at the first term A_1, d is added $n - 1$ times.	the 100^{th} term of 50, 48, 46, 44, . . . $d =$ $A_n =$ $A_{100} =$
Geometric	$A_n = A_1 r^{n-1}$	Each term is r times the term before it. Starting at the first term A_1, r is multiplied in $n - 1$ times.	the 15^{th} term of 27, 45, 75, 125, . . . $r =$ $A_n =$ $A_{15} =$

Arithmetic and Geometric Series

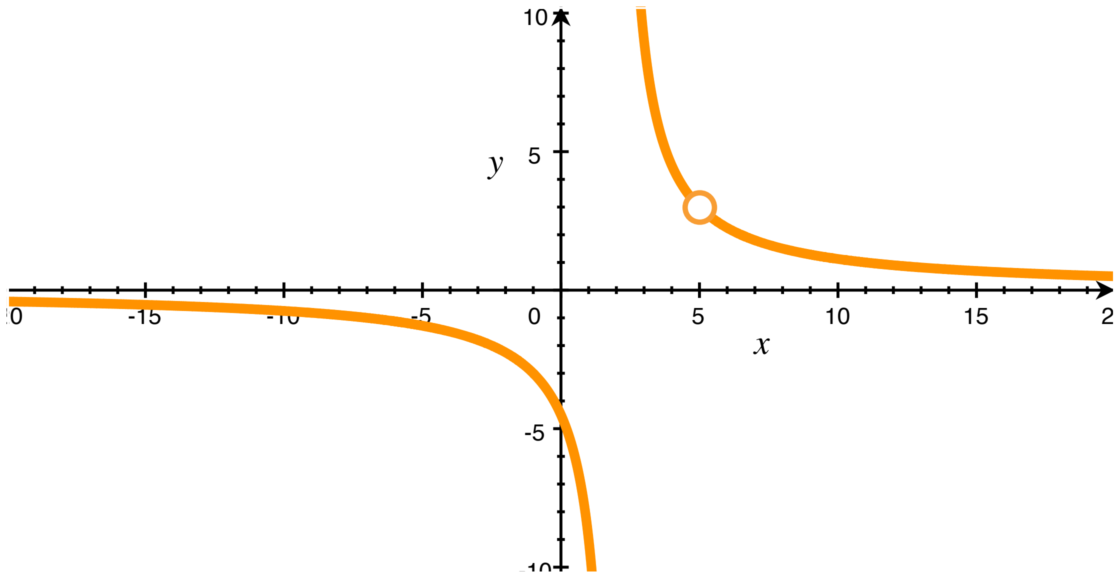
Type	Sum of n terms	Explanation	Example
Arithmetic	$S_n = n\left(\frac{A_1 + A_n}{2}\right)$	There are n terms. Because they are evenly spaced, every term on average is the average of the first and last term.	$20 + 23 + 26 + \ldots + 317 + 320$ $d =$ $320 =$, so $n =$ $S_{101} =$
Geometric	$S_n = A_1\left(\frac{1 - r^n}{1 - r}\right)$	Listing all the terms and dividing by $1 - r$ would yield $A_1(1 - r^n)$.	the first 20 terms of $48 + 72 + 108 + 162 + \ldots$ $r =$ $S_{20} =$
Infinite Geometric $n = \infty$ and $\lvert r \rvert < 1$	$S_n = \frac{A_1}{1 - r}$	If $\lvert r \rvert < 1$, the limit as n approaches infinity of r^n is zero. Use $r^n = 0$ in the geometric series formula.	$135 - 45 + 15 - 5 + \ldots$ $r =$ $S_\infty =$

Discontinuities and Limits

A **discontinuity** of a function is a point at which there is a break in the graph of the function.

The **limit** of a function as x approaches a is the value the function would equal if there were no discontinuity at $x = a$. If this value would be undfined, the limit is said not to exist.

The examples below refer to the function $f(x) = \frac{9x - 45}{x^2 - 7x + 10}$, graphed at right.



a	Graph at $x = a$	$f(a)$	$\lim_{x \to a} f(x)$
1	continuous		
2	discontinuous		
5	discontinuous		

Calculating Limits

Either method below can be used to find the limit of a rational function. If they do not work, this indicates that the limit does not exist.

Approach	Procedure	$\lim_{x \to 3} \frac{x^2 - 2x - 3}{2x^2 + 4x - 30}$
Calculate values approaching the limit	Plug in values closer and closer to $x = a$ until a pattern emerges.	
Cancel the factor causing the discontinuity	Factor $x - a$ out of the numerator and denominator.	

Slope Formulas

The slope between two points (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$. This is fine for a line, but on a curve the slope is different at every point.

To find the slope of a curve exactly at a specific point, the denominator $h = x_2 - x_1$ (the horizontal distance between the points) would have to be zero. This is not possible, but using calculus we can calculate the limit as h approaches zero. The calculus notation "$f'(x)$" is read "f prime of x".

Concept	Slope Equation	Explanation
Slope is rise over run	$m = \frac{y_2 - y_1}{x_2 - x_1}$	Slope is amount of vertical change divided by amount of horizontal change.
$f(x) = y$	$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$	Using function notation allows the concepts below.
$h = x_2 - x_1$	$f'(x) = \frac{f(x+h) - f(x)}{h}$	Let h be the horizontal distance between the points. This makes the first point $(x, f(x))$ and the second point $(x+h, f(x+h))$.
The exact slope is the limit as h approaches zero	$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$	h cannot actually be zero, but the closer it is to zero, the more precise the slope calculation is for that point.

Slope of a function

The formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ can be used to find the slope of any function at any point.

The example below shows finding the slope of $f(x) = x^3$ at $x = 5$.

Step	Equation
Identify $f(x+h)$	
Expand $(x+h)^3$	
Put $f(x+h)$ expression into formula	
Cancel x^3 term with $-x^3$ term	
Cancel h out of each term	
Plug in $h = 0$	
Plug in $x = 5$	

Slope formulas can be multiplied and added, as in the examples below.

Function	Slope formula
$f(x) = 4x^3$	
$f(x) = 4x^3 + 2x$	

Derivatives

The derivative of a function is its rate of change.

Original function	Units	Derivative function	Also known as	Units
$a(x)$ = amount of water in a pool	liters			
$b(x)$ = wage	dollars per hour			
$c(x)$ = distance fallen	meters			
$c'(x)$ = velocity	meters per second			
$d(x) = y$	n/a			

* In Calculus you will see the notation $\frac{\delta y}{\delta x}$, where δ (the Greek letter *delta*) means "change in".

Derivatives of common functions

The derivative function for any power function $f(x) = x^n$ is $f'(x) = nx^{n-1}$.

Original Function	Derivative Function	Explanation
$f(x) = 1$		The slope of a horizontal line is 0 at all points.
$f(x) = x$		The slope of the line $y = x$ is 1 at all points.
$f(x) = x^2$		At any point on the parabola $y = x^2$, the slope is double the x value.
$f(x) = x^3$		At any point on the curve $y = x^3$, the slope is triple the square of the x value.

Most functions have derivative functions, such as the common ones shown below.

Original Function	Derivative Function	At any value of x, the slope is equal to…
$f(x) = \sin x$		the cosine of the x value
$f(x) = \cos x$		the negative of the sine of the x value
$f(x) = \tan x$		the square of the secant of the x value
$f(x) = e^x$		the y value
$f(x) = \ln x$		the reciprocal of the x value