Chapter One • Due Monday, December 12

Functions and their Graphs

Functions

Domain and Range

Composition and Inverses

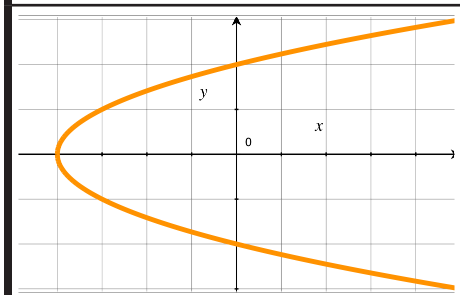
Calculator Input and Output

Transformations

Quadratics

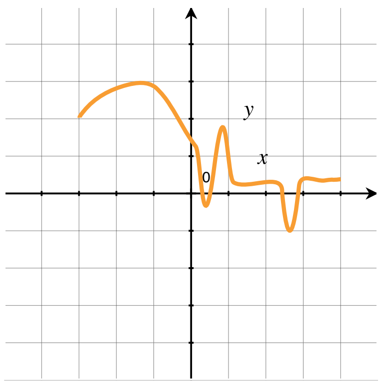
Functions

A **function** yields a specific output (value of the dependent variable) for every possible input (value of the independent variable, or **argument**). Even a single example of a single input having two different outputs means a relation is not a function.

Example	Independent Variable	Dependent Variable	Function?
$A = \pi r^2$			
$d(w) = 7w$			
{ (1, 4), (2, 5), (2, 9) }			
y = holiday in month x			
			

Domain and Range

The **domain** of a function is every value that could be plugged in, and the **range** is every value that could result.



The domain of the graph at right is limited to x values between -3 and 4. $f(5)$, for example, has no value.

The range is limited to y values between -1 and 3. Nowhere on the graph does $f(x)$ equal 4, for example.

Type of Function	Restrictions	Example	Domain of Example
Polynomial	none	$a(x) = 5x^2 + 2x$	
Rational	denominator must not be zero	$b(x) = \frac{x+5}{2x-6}$	
Even Roots	argument must be greater than or equal to zero	$c(x) = \sqrt{2x-6}$	
Logarithms	argument must be greater than zero	$d(x) = \log(2x-6)$	
Real-World	argument must make sense	$e(x)$ = number of high school students in grade x	

Composition

Compostion of functions is plugging an entire function or its value into another function.

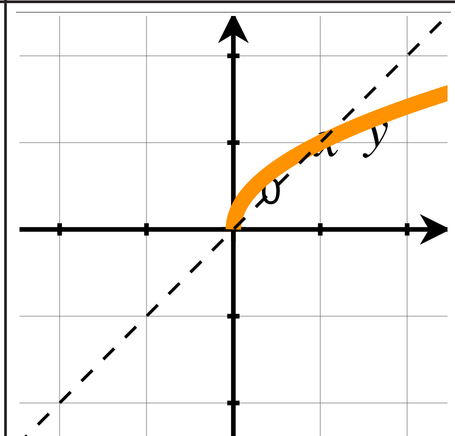
The examples below use the functions $a(x) = 4x^2 + 2x - 5$ and $b(x) = x + 3$.

Function Argument	Procedure	Example	Result
function value	Find the inner function value, and plug it into the outer function.	$a(b(2))$	
function expression	Replace the independent variable (each time) with the expression being plugged in.	$a(b(x))$	
composition	Do as above multiple times, working from inside to outside.	$b(b(a(b(2))))$	

$a(b(x))$ can also be written $(a \circ b)(x)$.

Inverses

If a function's independent and dependent variables are switched, the new relation is the **inverse** of the original. If it too is a function, it undoes the original function, and is labeled by $^{-1}$, such as $a^{-1}(x)$.

Representation	Way to find inverse	Example	Inverse of example
single operation	Identify the opposite of the operation.	$a(x) = x + 5$	
algebraic	Switch x and y, and solve for the new y.	$b(x) = 2x + 5$	
words	Switch the variables.	$c(x) =$ the number of minutes in x hours	
points	For each point, switch x with y.	{ (1, 4), (0, 9), (2, -5) }	
graph	Rotate the graph across $y = x$ diagonal.		

Parentheses

All numerators, denominators, negatives, and arguments have parentheses around them, whether or not they are written.

Reason	Example	Correct	Incorrect
Numerator	$\frac{4+8}{2}$		$4+8/2=4+\frac{8}{2}=8$
Denominator	$\frac{24}{8+4}$		$24/8+4=\frac{24}{8}+4=7$
Negative	$f(-5)$, given $f(x)=x^2$		$-5^2=-25$
Argument	$\sqrt{3+1}+5$		$\sqrt{}(3+1+5)=\sqrt{}9=3$

Significant Figures

The number of **significant figures** in a value is how many digits were measured or calculated. In normal contexts, nonzeros are always significant, but zeros are not sigifnicant if they are at the end and there is no decimal point or if they are at the beginning and there is a decimal point.

Be careful with calculator scientific notation. In the last example, 3.82×10^{-4} is the same as .000382, but it is very different from 3.82!

Value	Number of significant figures	Rounded to 3 SF
32,490		
.032490		
3.817007260E-4		

In special contexts, significant figures should be counted differently.

Value and Context	Number of significant figures	Reason
"exactly 3000 meters"		The distance was measured exactly, down to the meter.
growth factor of 1.025		The growth rate is .025.
decay factor of .998		The decay rate is .002.

Rounding

In addition to rounding up ending digits of 5, 6, 7, 8, and 9, keep the following guidelines in mind.

Guideline	Specifics	Example	Answers
Use exact values when possible.	When exact values are known, do not round them.	Given $f(x) = x^{20}$, evaluate $f(^4/_3)$.	
Use consistent decimal places appropriate for the context.	Use an appropriate, consistent number of decimal places for each answer within a specific context with the same units.	Add 8% tax to a $6 item and to a $10 item.	
Use enough significant figures.	Three significant figures are usually good for a final answer, but use more for the work. Avoid rounding to a single digit.	Convert 17 inches to feet.	
Don't use digits that aren't significant.	Don't use more significant figures in the answer than were originally given.	Convert 1.5 pounds to grams.	
Be careful with growth factors and decay factors.	When multiplying by $1 + r$, count the significant figures in r, not in $1 + r$.	What growth factor takes 5 years to yield an increase of 7.32%?	

Transformations

Transformation of a function changes its graph and its corresponding equation. In all contexts, anything done directly to x makes a horizontal change, and anything done directly to y makes a vertical change. See PreCalculus Chart 1 for detailed examples.

Transformation	Vertical	Horizontal	Change in graph
Translation			Each point is moved k units up or h units to the right.
Stretch			Each point is a times as far from the x-axis or b times as far from the y-axis.
Reflection			Each point is reflected to the other side of the x-axis or y-axis.

Quadratics

A **quadratic** equation is one in which the highest power is 2. Its graph is a parabola, the tip of which is called the **vertex**.

Form	Equation	To find the vertex	To find the *x*-intercepts
vertex	$y = a(x-h)^2 + k$		
standard	$y = ax^2 + bx + c$		
intercept	$y = a(x-p)(x-q)$		

Solving Quadratic Equations

Quadratic equations can be solved in multiple ways. The quadratic formula will work in all situations, but in some cases factoring or completing the square is simpler.

Method	Procedure	$x^2 - 4x + 3 = 15$	Most common error
factor	Set one side equal to zero, factor the other side, and set each factor equal to zero.		not setting one side equal to zero before factoring: $(x - 1)(x - 3) = 15$
complete the square	Make one side a perfect square and make the other side a number, so that the square root can be taken of each side.		not using $\pm$ to show both square roots: $x + 2 = 4$
quadratic formula	Put the equation in standard form equal to zero, and plug *a*, *b*, and *c* into the quadratic formula.		not using parentheses for *b* if it is negative: $x = \frac{-(-4) \pm \sqrt{-4^2 - 4(1)(-12)}}{2(1)}$

Chapter Two • Due Tuesday, December 13

Trigonometric Functions

Angles in Right Triangles

Angles in Circles

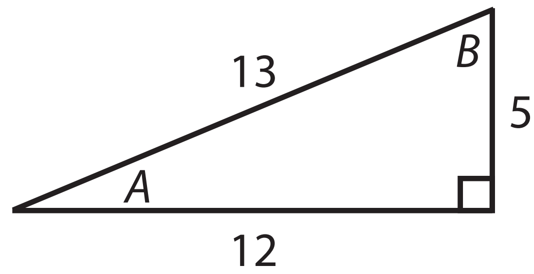
Radians

The Unit Circle

Trigonometric Identities

Trigonometric Functions in Right Triangles

There are six possible ratios that can be made using the lengths of two of the sides of a right triangle, and each is a trigonometric function of the angle. For example, sine is the length of the side opposite the angle divided by the length of the hypotenuse. If the other angle is used, the opposite side becomes the adjacent side, and vice versa.

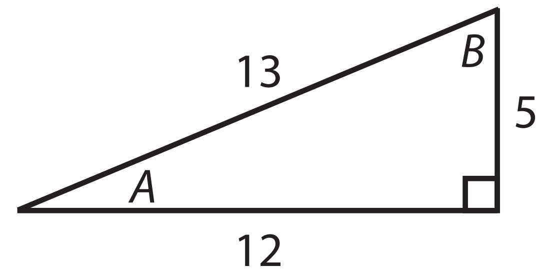


Function	Ratio	Reciprocal	Example *A*	Example *B*
sine				
cosine				
tangent				
cotangent				
secant				
cosecant				

Inverse Trigonometric Functions

Trigonometric functions find the ratio of sides made by a given angle.

Inverse trigonometric functions find the angle needed to make a given ratio of sides.

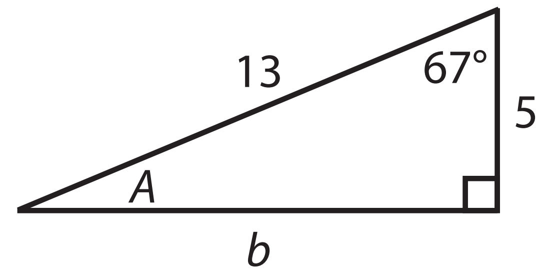


Trigonometric functions are used to find lengths of sides, and inverse trigonometric functions are used to find measures of angles.

Inverse Function	Notation	Example *A*	Example *B*
sine inverse			
cosine inverse			
tangent inverse			

Solving a Right Triangle

Trig equations involve three values: two side lengths and an angle measure. If any two of these values are known, they can be plugged in to solve for the third.



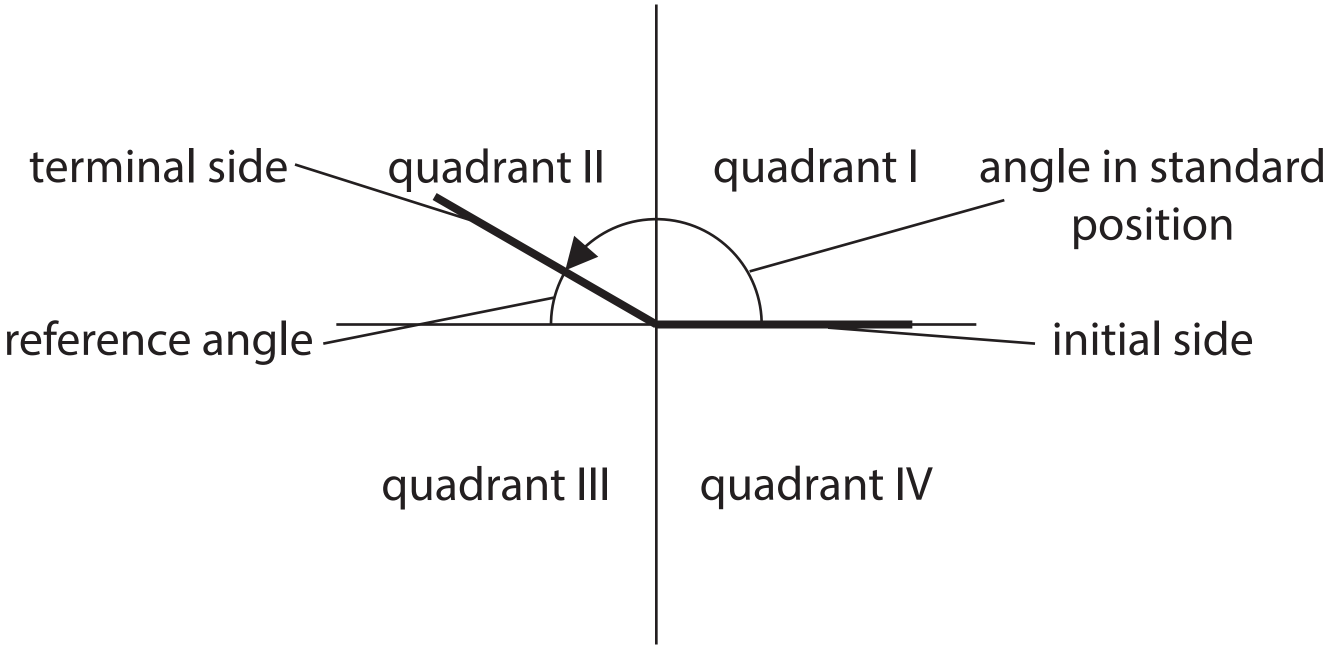
In many cases, more than one trig function can be used. For example, b could also be found by using tangent: $\tan 67° = \frac{b}{5}$.

Part solving for	Method	Sine example
side	Multiply each side of the trig equation by the denominator, and divide by the trig function if necessary. Then evaluate on a calculator.	
angle	Do the applicable inverse trig function on each side, to make one side just the angle. Evaluate its value on a calculator.	

Angle Terminology

The terms below are helpful in describing and sketching angles.

Term	Definition
Initial Side	
Terminal Side	
Standard Position	
Quadrant	
Reference Angle	

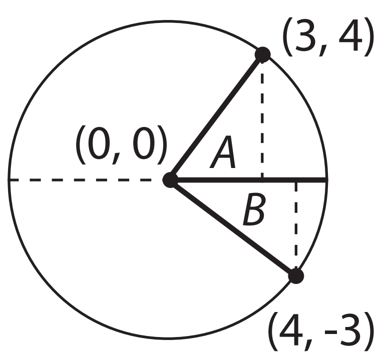


Trigonometric Functions in Circles

The trig function definitions earlier were for angles in right triangles, and thus only apply to angles between 0° and 90°. An expanded definition, that applies to all angles, is based on placing the angle in standard position within a circle centered at (0, 0). The *x* and *y* coordinates of the point where the terminal side intersects the circle are used instead of adjacent and opposite, respectively, and the radius of the circle (which can be found by the Pythagorean theorem) is used instead of hypotenuse.

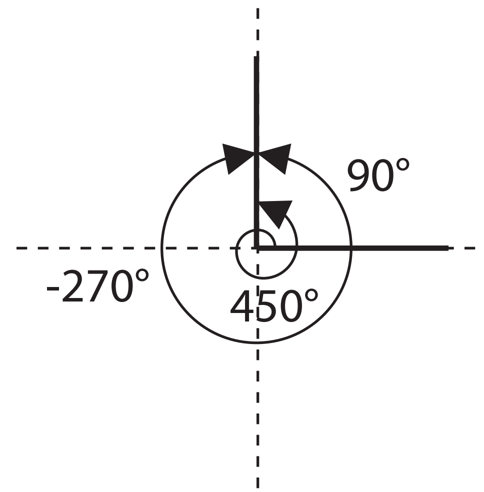
Drawing a vertical line from the point to the *x*-axis makes a right triangle. If the angle is between 0° and 90°, then the definitions are exactly same; otherwise it is the same except that *x*, *y*, or both, are negative.

Function	Ratio	Example *A*	Example *B*
sine			
cosine			
tangent			
cotangent			
secant			
cosecant			



Coterminal Angles

The measure of an angle can be any size, including negative. An angle not between 0° and 360° has the same terminal side as an angle that is between 0° and 360°, and looks the same. Such identical-looking angles are called **coterminal**, and can make trigonometric problems much simpler. They can be found by adding or subtracting 360° any number of times.

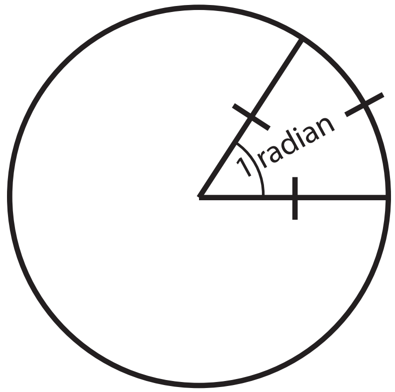


In the example at right, the 90° angle (going up from the *x*-axis), the -270° angle (going down from the *x*-axis), and the 450° angle (going up from the *x*-axis a full circle and then another 90°) are all coterminal.

Angle example	Next three bigger coterminal angles	Next three smaller coterminal angles	Smallest positive coterminal angle
90°			
1470°			

Radians

A **radian** is the measure of an angle in a circle such that the radius of the circle and the arc of the circle are the same length.

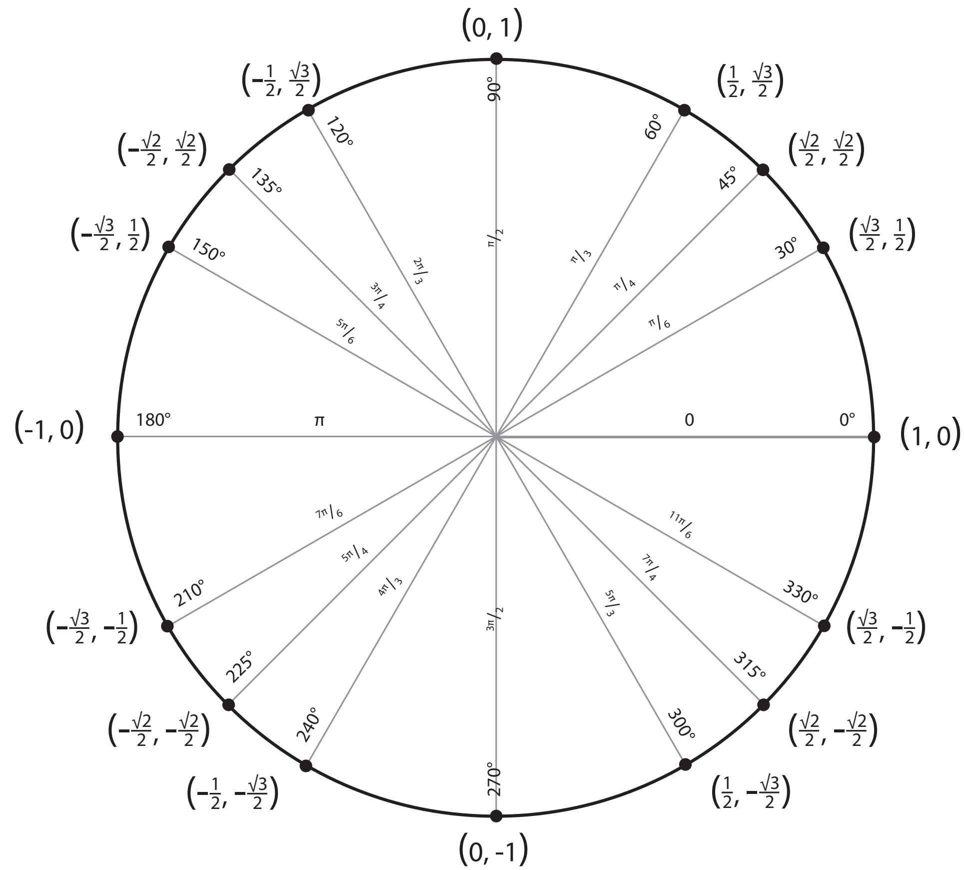


Since the arc length of a full circle (360°) is its circumference ($C = 2\pi r$), **$360° = 2\pi$ radians**. Therefore, radians can be converted to degrees by multiplying by $\frac{360°}{2\pi}$. A simpler method is to apply the conversions in the table below. For example, $\frac{5\pi}{12} = 5(\frac{\pi}{12}) = 5(15°) = 75°$.

Measure in radians	Measure in degrees	Example (x5)	Example (x11)
$\frac{\pi}{12}$			
$\frac{\pi}{6}$			
$\frac{\pi}{4}$			
$\frac{\pi}{3}$			
$\frac{\pi}{2}$			
π			

The Unit Circle

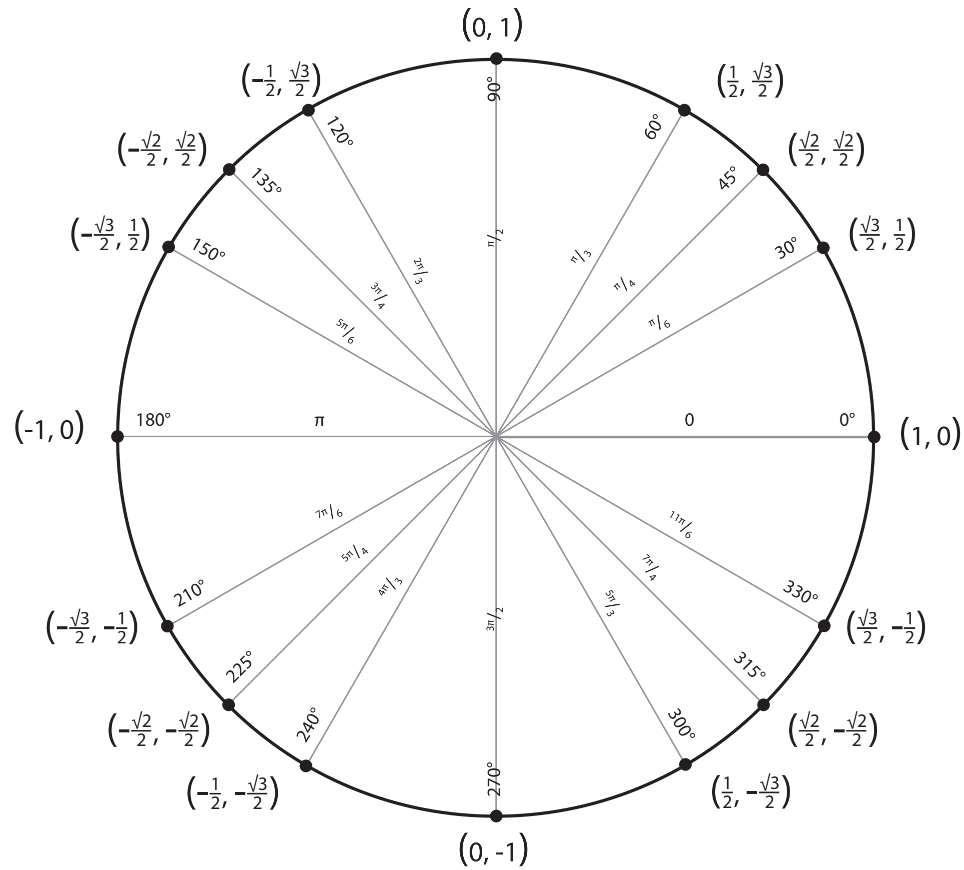
A unit circle is a circle with a radius of 1. They are usually marked in increments of $\frac{\pi}{6}$ (30°) and $\frac{\pi}{4}$ (45°), and can be used to find trig values for these angles, based on the definitions of trig functions in a circle.



Function	Definition	In unit circle	Example: 120°
Sine	$\sin\theta = \frac{y}{r}$		
Cosine	$\cos\theta = \frac{x}{r}$		
Tangent	$\tan\theta = \frac{y}{x}$		

Inverse Functions and the Unit Circle

Values of inverse trig functions can be found on the unit circle. However, since they are functions, they can only have one output, so their outputs are limited to a 180° range, as shown below. For example, even though 30° and 150° both have a sine of ½, $\sin^{-1} \frac{1}{2} = 30°$ only.



Function	Angle	Range	Example
$\sin^{-1} b$	where $y = b$		
$\cos^{-1} b$	where $x = b$		
$\tan^{-1} b$	where $\frac{y}{x} = b$		

Trigonometric Identities

An **identity** is an equation that is always true because the two sides are algebraically equivalent, such as $a + b = a - (\text{-b})$. Common basic trig identities are shown below.

Reciprocal Identity	Proof
	$\frac{y}{r} = 1 \div \frac{r}{y}$
	$\frac{x}{r} = 1 \div \frac{r}{x}$
	$\frac{y}{x} = 1 \div \frac{x}{y}$
	$\frac{x}{y} = 1 \div \frac{y}{x}$
	$\frac{r}{x} = 1 \div \frac{x}{r}$
	$\frac{r}{y} = 1 \div \frac{y}{r}$

Quotient Identity	Proof
	$\frac{y}{x} = \frac{y \div r}{x \div r}$
	$\frac{x}{y} = \frac{x \div r}{y \div r}$

Pythagorean Identity	Proof
	$(\frac{y}{r})^2 + (\frac{x}{r})^2 = \frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2}$
	$(\frac{r}{x})^2 - (\frac{y}{x})^2 = \frac{r^2 - y^2}{x^2} = \frac{x^2}{x^2}$
	$(\frac{r}{y})^2 - (\frac{x}{y})^2 = \frac{r^2 - x^2}{y^2} = \frac{y^2}{y^2}$
	see above
	see above
	see above
	see above
	see above
	see above

Double Angle Identities		

Verifying Trigonometric Identities

Verifying an identity means establishing that an equation is in fact an identity. This is done by rewriting one side of the equation until it is written the same as the other side. In the example below, the identity $\cot x + \tan x = \csc x \sec x$ is verified.

$\cot x + \tan x$	Justification
	Rewrite using quotient identities.
	Multiply each term by 1 to get a common denominator.
	Simplify.
	Add.
	Rewrite the numerator using a Pythagorean identity.
	Rewrite as separate factors.
$= \csc x \sec x$	Rewrite using reciprocal identities.

Techniques for Verifying Trigonometric Identities

Identities can be verified using basic arithmetic and algebra along with the identities given earlier. There is usually not one specific way to verify a given identity. Commonly, an approach will be tried and then discarded in favor of another.

Several common techniques are listed below. See PreCalculus Chart 2 for details and examples.

Technique	Summary
Rewrite a trigonometric expression without fractions.	Use a reciprocal identity or a quotient identity.
Rewrite a trigonometric expression using only sine and cosine.	Use a reciprocal identity or a quotient identity.
Rewrite a trigonometric expression using a Pythagorean identity.	Use a Pythagorean identity.
Split a fraction with multiple terms in the numerator into separate fractions.	Make a separate fraction for each term in the numerator, using the same denominator.
Add or subtract terms when both are fractions.	Use a common denominator.
Use a conjugate to simplify a trigonometric fraction.	Multiply by a conjugate to get a Pythagorean identity.
Factor a trigonometric expression.	Factor into factors that can be rewritten using identities.

Chapter Three • Due Thursday, December 15

Trigonometric Equations

Solving Simple Trigonometric Equations Algebraically

Solving Complicated Trigonometric Equations Algebraically

Graphs of Sine and Cosine Functions

Solving Trigonometric Equations Graphically

Two Solutions to Simple Trigonometric Equations

Based on the symmetry of the unit circle, simple trig equations such as sin $A = ½$ have two solutions between 0° and 360°. If not labeled on the unit circle, these can be found by using an inverse function and then finding a second solution as shown below.

Function	1st Solution	2nd Solution	Reason	Sketch
sine	$A_1 = \sin^{-1} b$		Subtracting from 180° reflects the terminal side across the y-axis, keeping the same y-coordinate.	
cosine	$A_1 = \cos^{-1} b$		Subtracting from 360° reflects the terminal side across the x-axis, keeping the same x-coordinate.	
tangent	$A_1 = \tan^{-1} b$		Adding 180° rotates the terminal side to the opposite quadrant, keeping the same x-coordinate and y-coordinate. (The sign of each coordinate switches, but since they both switch, this has no effect.)	

Solving Trigonometric Equations

Solving more complicated trigonometric equations involves algebra and trigonometric identities.

Step	$5 \sec^2 3x = 10$	Notes for example
Isolate the trig function.		Divide each side by 5.
If cot x, sec x, or csc x is isolated, take the reciprocal.		The reciprocal of sec x is cos x. (The argument is unchanged.)
If the trig function is squared, take the sqaure root.		Don't forget to put $\pm$. $\sqrt{½}$ can be simplified to $\frac{\sqrt{2}}{2}$ and.
Apply the inverse trig function.		Make sure to do both if there is a $\pm$.
Find a second solution for each solution found.		Subtract the $\cos^{-1}$ results from 360° (see previous slide).
Add $360°n$ (or $2\pi n$) to each solution.		This finds coterminal solutions.
Use algebra to solve.		Divide each side by 3 to solve for x. Don't forget to divide the $360°n$.

Solving Trigonometric Equations by Factoring

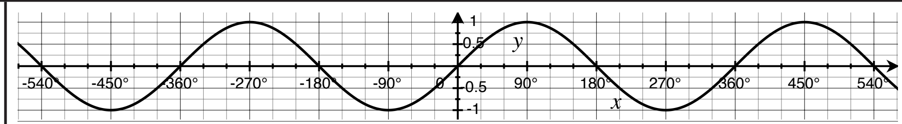
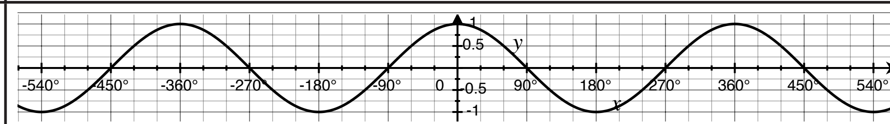
In many cases, trigonometric equations can be solved most easily by factoring, like polynomial equations.

Step	$x^3 + x^2 = 20x$	$\tan^3 x + \tan^2 x = 20 \tan x$
Set the equation equal to 0.	$x^3 + x^2 - 20x = 0$	
Factor the expression.	$x(x^2 + x - 20) = 0$ $x(x - 4)(x + 5) = 0$	
Set each factor equal to zero.	$x = 0$ $x - 4 = 0$ $x + 5 = 0$	
Solve each equation.	$x = 0$ $x = 4$ $x = -5$	
Find additional solutions.*	n/a	

* For tangent (which is most common in trig factoring problems), additional solutions can be found by simply adding $180°n$ to each original solution.

Graphs of Sine and Cosine

The graph of $y = \sin x$ is a wave that, because of coterminal angles, repeats itself every 360°. The same is true for $y = \cos x$.

Function	At *y*-axis	Graph
$y = \sin x$	at middle of wave, going up	
$y = \cos x$	at top of wave, going down	

The graph of $y = \cos x$ is the same as the graph of $y = \sin x$, except that it is translated to the left by 90°: $\cos x = \sin (x + 90°)$. Likewise, $\sin x = \cos (x - 90°)$.

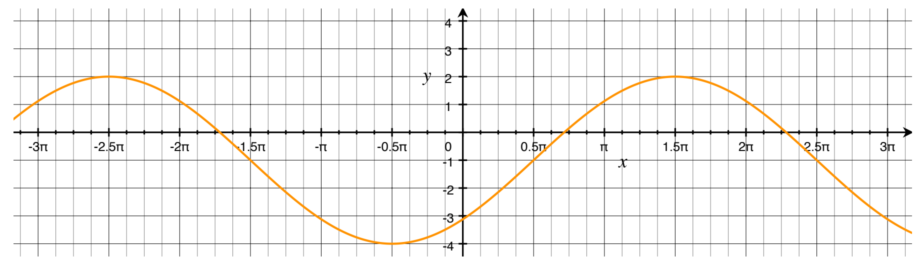
Transformations of Sine and Cosine

Like any function, sine and cosine can be transformed by translation, stretch, and reflection (see 1-E). For translations and stretches, there is terminology specific to trig functions.

Aspect	Equation	Transformation	Graphic example
Amplitude		vertical stretch of a	$y = 2 \sin x$
Period		horizontal stretch of $^1/_b$	$y = \sin 2x$
Phase shift		horizontal translation of c	$y = \sin (x - 90°)$
Vertical shift		vertical translation of d	$y = 2 + \sin x$

Equations of Sine and Cosine Graphs

The equation of a sine graph $y = d + a \sin b(x - c)$ can be deterimined by identifying the parameters *a*, *b*, *c*, and *d*, based on the amplitude, period, horizontal shift, and vertical shift of the graph.

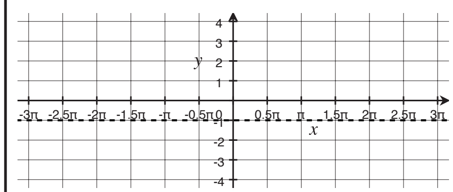
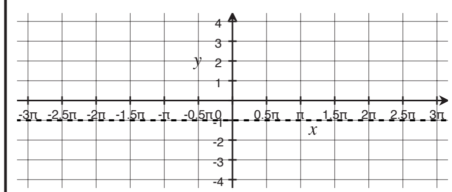
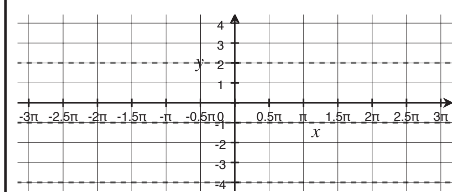
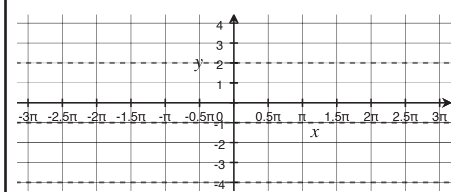
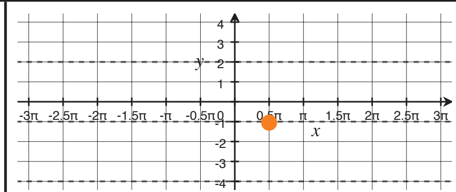
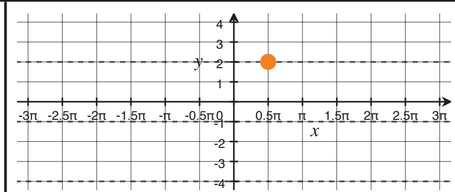
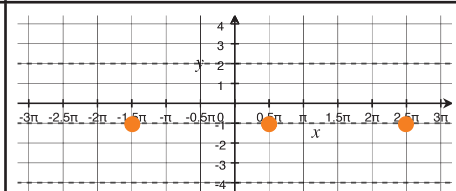
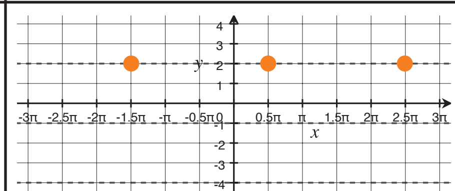
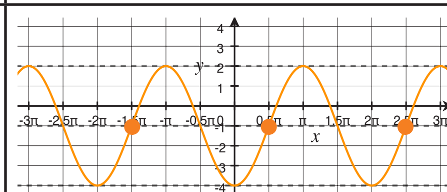
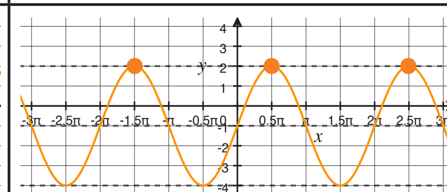


The equation of the graph above can be written $y = -1 + 3 \sin \frac{1}{2}(x - \frac{\pi}{2})$ or $y = -1 + 3 \cos \frac{1}{2}(x - \frac{3\pi}{2})$.

Parameter	How to identify	Sine example	Cosine Example
d	average of the the highest and lowest *y* value		same
a	distance from *d* to the top of the curve		same
b	$b = \frac{2\pi}{\text{period}}$ (or $b = \frac{360°}{\text{period}}$), where the period is how far the graph goes before repeating, such as from one peak to the next		same
c	the distance from the *y*-axis to where the graph crosses the line $y = d$ going up (for sine) or where the graph peaks (for cosine)		

Sketches of Sine and Cosine graphs

A sine graph or cosine graph can be sketched by identifying *a*, *b*, *c*, and *d*, and applying them as shown below.

Aspect	Procedure	$y = -1 + 3\sin\left(x - \frac{\pi}{2}\right)$	$y = -1 + 3\cos\left(x - \frac{\pi}{2}\right)$
vertical shift	The middle of the curve is at $y = d$.		
amplitude	The top of the curve is at $y = d + a$, and the bottom is at $y = d - a$.		
phase shift	For sine, plot a point at (c, d), or for cosine, plot a point $(c, d + a)$.		
period	The period is $\frac{2\pi}{b}$. Plot a point this far to the right of the first point plotted, and another point this far to the left.		
curve	Sketch a sine or cosine curve from each point to the next. If *a* is negative, sketch it upside-down.		

Chapter Four • Due Monday, December 19

Nonright Triangles

The Law of Sines

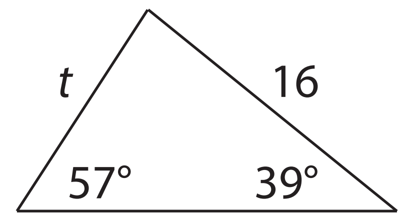
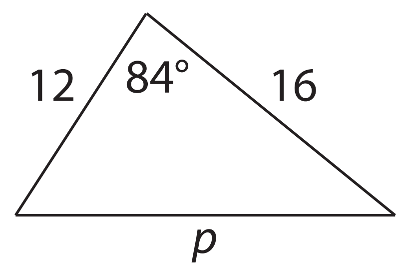
The Law of Cosines

Areas of Triangles

Solving Nonright Triangles

In nonright triangles, there is no hypotenuse, and each angle has two adjacent sides, making the original definitions of sine, cosine, and tangent (in terms of adjacent, opposite, and hypotenuse) meaningless.

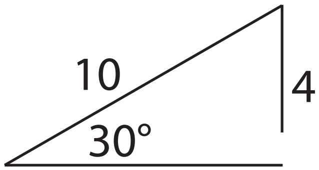
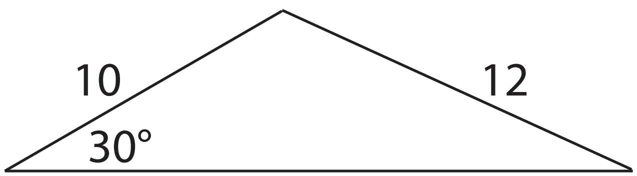
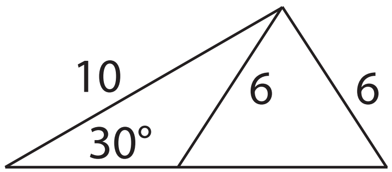
Instead, sides and angles in nonright triangles can be found with the **law of sines** or the **law of cosines**.

Method	When used	Equation	Example
Law of Sines	An angle and the side opposite it are known, and so is one other side or angle.	$\frac{a}{\sin A} = \frac{b}{\sin B}$	
Law of Cosines	Two sides are known, and so is the angle between them or the side between them.	$c^2 = a^2 + b^2 - 2ab\cos C$	

The law of cosines can be algebraically rewritten as $c = \sqrt{a^2 + b^2 - 2ab\cos C}$ or as $C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$ to make it easier to calculate a side or an angle, respectively.

The Ambiguous Case of the Law of Sines

When solving for an angle using an inverse trig function, there are infinitely many solutions. For the law of sines, there may be zero, one, or two such angles that fit within the given triangle.

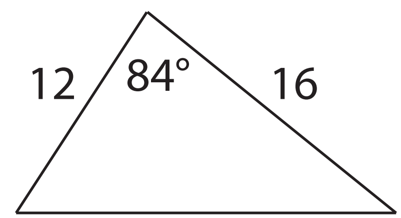
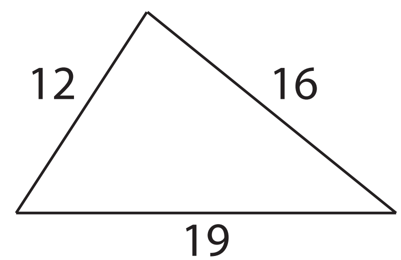
Solutions	When it happens	Example
zero	The side opposite the known angle is too short to connect the other two sides.	
one	The side opposite the known angle is longer than the other known side.	
two	The side opposite the known angle is shorter than the other known side, but still long enough to reach the unknown side.	

Areas of Triangles

The three **altitudes** of a triangle are the line segments from each vertex to the opposite side, connected at a right angle. Their lengths can be calculated for ΔABC as $h = b \sin C$, $h = c \sin A$, and $h = a \sin B$.

Since an altitude is a height and the side perpendicular to it is a base, the area of a triangle can be calculated using these two values.

If only the side lengths of a triangle are known, its area can be calculated by Hero's formula, which is based on the law of cosines and involves the semiperimeter $s = \frac{a+b+c}{2}$.

Method	Area	Example
½ base • height	$\frac{1}{2}ab \sin C$	
Hero's formula	$\sqrt{s(s-a)(s-b)(s-c)}$	

If the angles and only one side are known, a second side can be found using the law of sines.