Chapter Nine

Derivatives

Sequences and Series

Limits

Slopes of Curves

Derivatives

Summation Notation

The $\sum$ symbol is used for summation. Every integer from a starting number (at the bottom) to an ending number (at the top) is plugged into the expression.

The list of resulting values is a **sequence**.

The summation of the resulting values is a **series**, which is what $\sum$ notation represents.

Example	Numbers plugged in	Sequence	Series
$\sum_{x=1}^{4} 10x$	$x = 1, 2, 3, 4$	$A_n = 10, 20, 30, 40$	$S_n = 10 + 20 + 30 + 40$
$\sum_{x=6}^{8} (x^2 + 2x - 10)$	$x = 6, 7, 8$	$A_n = 38, 53, 70$	$S_n = 38 + 53 + 70$
$\sum_{x=60}^{100} (2x + 1)$	$x = 60, 61, 62, \ldots, 99, 100$	$A_n = 121, 123, 125, \ldots, 199, 201$	$S_n = 121 + 123 + 125 + \ldots + 199 + 201$

Two common types of sequences and series are arithmetic and geometric.

Arithmetic sequences have a constant difference d from each term to the next.

Geometric sequences have a constant ratio r from each term to the next.

Arithmetic and Geometric Sequences

Type	n^{th} term	Explanation	Example
Arithmetic	$A_n = A_1 + (n-1)d$	Each term is d more than the term before it. Starting at the first term A_1, d is added $n-1$ times.	the 100^{th} term of 50, 48, 46, 44, . . . $d = -2$ $A_n = 50 + (n-1)(-2)$ $A_{100} = 50 + (99)(-2) =$ **-148**
Geometric	$A_n = A_1 r^{n-1}$	Each term is r times the term before it. Starting at the first term A_1, r is multiplied in $n-1$ times.	the 15^{th} term of 27, 45, 75, 125, . . . $r = \frac{5}{3}$ $A_n = 27(\frac{5}{3})^{n-1}$ $A_{15} = 27(\frac{5}{3})^{14} \approx$ **34,455**

Arithmetic and Geometric Series

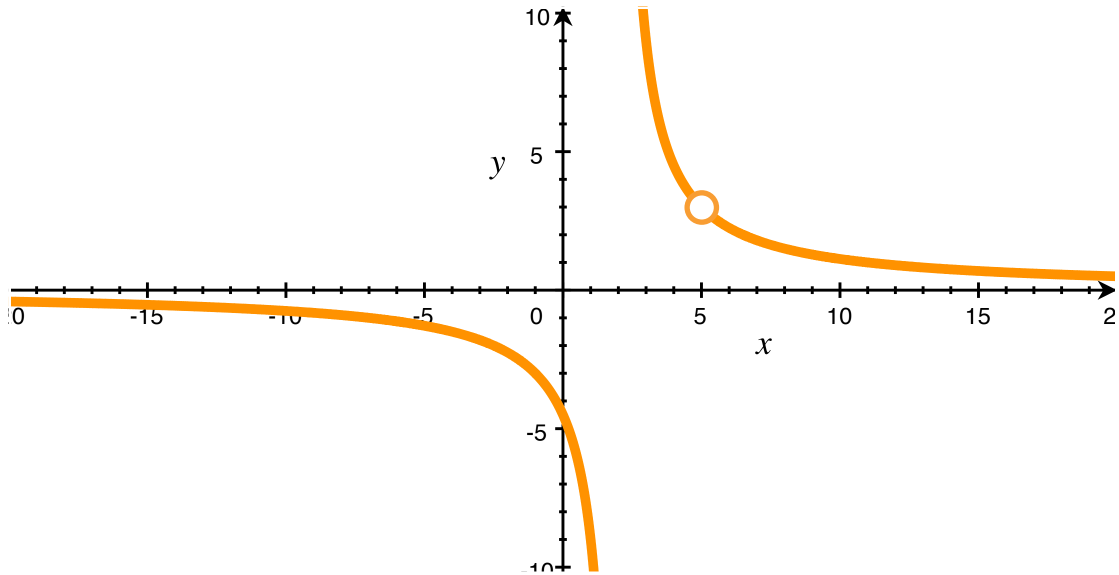
Type	Sum of n terms	Explanation	Example
Arithmetic	$S_n = n\left(\frac{A_1 + A_n}{2}\right)$	There are n terms. Because they are evenly spaced, every term on average is the average of the first and last term.	$20 + 23 + 26 + \ldots + 317 + 320$ $d = 3$ $320 = 20 + (n - 1)3$, so $n = 101$ $S_{101} = 101\left(\frac{20 + 320}{2}\right) = 17170$
Geometric	$S_n = A_1\left(\frac{1 - r^n}{1 - r}\right)$	Listing all the terms and dividing by $1 - r$ would yield $A_1(1 - r^n)$.	the first 20 terms of $48 + 72 + 108 + 162 + \ldots$ $r = \frac{3}{2}$ $S_{20} = 48\left(\frac{1 - (3/2)^{20}}{1 - 3/2}\right) \approx 319{,}129$
Infinite Geometric $n = \infty$ and $\lvert r \rvert < 1$	$S_n = \frac{A_1}{1 - r}$	If $\lvert r \rvert < 1$, the limit as n approaches infinity of r^n is zero. Use $r^n = 0$ in the geometric series formula.	$135 - 45 + 15 - 5 + \ldots$ $r = -\frac{1}{3}$ $S_\infty = \left(\frac{135}{1 - (-1/3)}\right) = 101.25$

Discontinuities and Limits

A **discontinuity** of a function is a point at which there is a break in the graph of the function.

The **limit** of a function as x approaches a is the value the function would equal if there were no discontinuity at $x = a$. If this value would be undfined, the limit is said not to exist.

The examples below refer to the function $f(x) = \frac{9x - 45}{x^2 - 7x + 10}$, graphed at right.



a	Graph at $x = a$	$f(a)$	$\lim_{x \to a} f(x)$
1	continuous	$f(1) = \frac{9 - 45}{1 - 7 + 10} = \frac{-36}{4} = -9$	$\lim_{x \to 1} \frac{9x - 45}{x^2 - 7x + 10} = -9$
2	discontinuous	$f(2) = \frac{18 - 45}{4 - 14 + 10} = \frac{-27}{0}$, which is **undefined**	$\lim_{x \to 2} \frac{9x - 45}{x^2 - 7x + 10}$ **does not exist**
5	discontinuous	$f(5) = \frac{45 - 45}{25 - 35 + 10} = \frac{0}{0}$, which is **undefined**	$\lim_{x \to 5} \frac{9x - 45}{x^2 - 7x + 10} = 3$

Calculating Limits

Either method below can be used to find the limit of a rational function. If they do not work, this indicates that the limit does not exist.

Approach	Procedure	$\lim_{x\to 3} \frac{x^2-2x-3}{2x^2+4x-30}$
Calculate values approaching the limit	Plug in values closer and closer to $x = a$ until a pattern emerges.	$f(3.1) = \frac{3.1^2 - 2(3.1) - 3}{2(3.1)^2 + 4(3.1) - 30} = 0.253$ $f(3.01) = \frac{3.01^2 - 2(3.01) - 3}{2(3.01)^2 + 4(3.01) - 30} = 0.2503$ $f(3.001) = \frac{3.001^2 - 2(3.001) - 3}{2(3.001)^2 + 4(3.001) - 30} = 0.25003$ $\lim_{x\to 3} \frac{x^2-2x-3}{2x^2+4x-30} = 0.25$
Cancel the factor causing the discontinuity	Factor $x - a$ out of the numerator and denominator.	$\frac{x^2-2x-3}{2x^2+4x-30} = \frac{(x+1)(x-3)}{2(x+5)(x-3)} = \frac{x+1}{2x+10}$ $\lim_{x\to 3} \frac{x^2-2x-3}{2x^2+4x-30} = \lim_{x\to 3} \frac{x+1}{2x+10} = \frac{4}{16} = \frac{1}{4}$

Slope Formulas

The slope between two points (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$. This is fine for a line, but on a curve the slope is different at every point.

To find the slope of a curve exactly at a specific point, the denominator $h = x_2 - x_1$ (the horizontal distance between the points) would have to be zero. This is not possible, but using calculus we can calculate the limit as h approaches zero. The calculus notation "$f'(x)$" is read "f prime of x".

Concept	Slope Equation	Explanation
Slope is rise over run	$m = \frac{y_2 - y_1}{x_2 - x_1}$	Slope is amount of vertical change divided by amount of horizontal change.
$f(x) = y$	$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$	Using function notation allows the concepts below.
$h = x_2 - x_1$	$f'(x) = \frac{f(x+h) - f(x)}{h}$	Let h be the horizontal distance between the points. This makes the first point $(x, f(x))$ and the second point $(x+h, f(x+h))$.
The exact slope is the limit as h approaches zero	$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$	h cannot actually be zero, but the closer it is to zero, the more precise the slope calculation is for that point.

Slope of a function

The formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ can be used to find the slope of any function at any point.

The example below shows finding the slope of $f(x) = x^3$ at $x = 5$.

Step	Equation
Identify $f(x+h)$	$f(x+h) = (x+h)^3$
Expand $(x+h)^3$	$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$
Put $f(x+h)$ expression into formula	$f'(x) = \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$
Cancel x^3 term with $-x^3$ term	$f'(x) = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$
Cancel h out of each term	$f'(x) = 3x^2 + 3xh + h^2$
Plug in $h = 0$	$f'(x) = 3x^2 + 0 + 0$
Plug in $x = 5$	$f'(5) = 3(5)^2 + 0 + 0 = 75$

Slope formulas can be multiplied and added, as in the examples below.

Function	Slope formula
$f(x) = 4x^3$	$f'(x) = 4(3x^2) = 12x^2$
$f(x) = 4x^3 + 2x$	$f'(x) = 12x^2 + 2$

Derivatives

The derivative of a function is its rate of change.

Original function	Units	Derivative function	Also known as	Units
$a(x) =$ amount of water in a pool	liters	$a'(x) =$ rate of change of water in pool	fill rate	liters per minute
$b(x) =$ wage	dollars per hour	$b'(x) =$ rate of change of wage	raises	dollars per hour per year
$c(x) =$ distance fallen	meters	$c'(x) =$ rate of change of distance fallen	velocity	meters per second
$c'(x) =$ velocity	meters per second	$c''(x) =$ rate of change of velocity	acceleration	meters per second per second
$d(x) = y$	n/a	$d'(x) =$ rate of change in y*	slope	n/a

* In Calculus you will see the notation $\frac{\delta y}{\delta x}$, where δ (the Greek letter *delta*) means "change in".

Derivatives of common functions

The derivative function for any power function $f(x) = x^n$ is $f'(x) = nx^{n-1}$.

Original Function	Derivative Function	Explanation
$f(x) = 1$	$f'(x) = 0$	The slope of a horizontal line is 0 at all points.
$f(x) = x$	$f'(x) = 1$	The slope of the line $y = x$ is 1 at all points.
$f(x) = x^2$	$f'(x) = 2x$	At any point on the parabola $y = x^2$, the slope is double the x value.
$f(x) = x^3$	$f'(x) = 3x^2$	At any point on the curve $y = x^3$, the slope is triple the square of the x value.

Most functions have derivative functions, such as the common ones shown below.

Original Function	Derivative Function	At any value of x, the slope is equal to…
$f(x) = \sin x$	$f'(x) = \cos x$	the cosine of the x value
$f(x) = \cos x$	$f'(x) = -\sin x$	the negative of the sine of the x value
$f(x) = \tan x$	$f'(x) = \sec^2 x$	the square of the secant of the x value
$f(x) = e^x$	$f'(x) = e^x$	the y value
$f(x) = \ln x$	$f'(x) = {}^1/_x$	the reciprocal of the x value