CHAPTER NINE: DERIVATIVES

Review May 16 ⚘ Test May 23

Calculus begins with the study of rates of change, called derivatives. For example, the derivative of velocity is acceleration because acceleration is the rate velocity is changing, and the derivative of a function is its slope because slope is the rate the value of the function is changing. Given a function f, a derivative function f′ can be found; f(x) yields the value of the function at the given value of x, and f′(x) yields the slope (that is, the rate of change) of the function at the given value of x.

9-A Sequences and Series — Thursday • 5/4

sequence • arithmetic • geometric • series • summation notation

1. Identify arithmetic and geometric sequences.
2. Find a formula for the n^{th} term of an arithmetic or geometric sequence.
3. Find the sum of a series by adding.
4. Write an arithmetic or geometric series in summation notation.
5. Count the number of terms in a series written in summation notation.
6. Find the sum of an arithmetic or geometric series.

9-B Limits — Tuesday • 5/9

limit • discontinuous • rational function

1. Find the limit of a simple function at a given value.
2. Use an example and the concept of limits to derive the formula for the sum of an infinite geometric series.
3. Use a graph to find the limit of a function f as it approaches a discontinuity.
4. Calculate values to estimate the limit of a function f as x approaches a number a.
5. Algebraically find the limit of a rational function f as x approaches a number a.

9-C Slopes of Curves — Thursday • 5/11

1. Calculate an estimate of the slope of a function at a given value of x.
2. Algebraically find the slope function of a power function, and use it to calculate the exact slope at a given value of x.
3. Calculate the exact slope of a polynomial function at a given value of x.

9-D Derivatives — Tuesday • 5/16

derivative • differentiation • power rule • tangent to a curve

1. State the deriviative function of a real-world function, using correct units.
2. Find the derivative of a polynomial function.
3. Use derivatives to find the slope of a function at a given point.
4. Find the equation of a line tangent to a curve at a given x-value a.

9-A Sequences and Series

A SEQUENCE is an ordered list.

The notation A_n represents the n^{th} term of sequence A. It means the same as function notation $A(n)$, but n must be an integer. Any letters can be used.

A_1 represents term #1 in sequence A. Sequences are often written so that they start at term #1.

An ARITHMETIC Sequence changes by a constant difference from each term to the next: $d = A_2 - A_1 = A_3 - A_2 = A_4 - A_3 = \ldots$.

A GEOMETRIC Sequence changes by a constant ratio from each term to the next: $r = A_2 \div A_1 = A_3 \div A_2 = A_4 \div A_3 = \ldots$.

❶ Identify arithmetic and geometric sequences.

1. From each term subtract the term before it. The sequence is arithmetic if this difference d is the same every time.
2. Divide each term by the term before it. The sequence is geometric if this ratio r is the same every time.

❶ What type of sequence is 5, -10, 20, -40, …?

$-10 - 5 = -15 \neq 20 - -10 = 30$, so the sequence is not arithmetic.

$-10 \div 5 = -2 = 20 \div -10 = -2 = -40 \div 20 = -2 = r$, so the sequence is geometric.

The n^{th} term of an arithmetic sequence A is $A_n = A_1 + d(n - 1)$.

The n^{th} term of a geometric sequence A is $A_n = A_1 r^{n-1}$.

❷ Find a formula for the n^{th} term of an arithmetic or geometric sequence.

1. Identify the constant difference d or the constant ratio r. Do not round r.
2. If A_1 is unknown, subtract d or divide by r as many times as needed to find it.
3. Plug d or r and the first term A_1 into the appropriate sequence formula above.

❷ Find an equation for the following sequences.

a) 49, 40, 31, 22, …

1. $d = 40 - 49 = 31 - 40 = 22 - 31 = -9$

3. $A_n = 49 - 9(n - 1)$, which simplifies to $A_n = -9n + 58$

b) 192, 144, 108, 81, …

$r = \frac{144}{192} = \frac{108}{144} = \frac{81}{108} = \frac{3}{4}$

$A_n = 192\left(\frac{3}{4}\right)^{n-1}$

A SERIES is the addition of terms in a sequence.

Series can be written in SUMMATION Notation with the Greek letter capital sigma, Σ. The sum of terms a through b of sequence A is $S_n = \sum_{n=a}^{b} A_n$.

❸ Find the sum of a series by adding.

1. Plug the number below Σ into the expression.
2. Repeat step 1, but increase the number being plugged in by 1.
3. Repeat step 2 until you have plugged in the number written above Σ.
4. Add all the resulting terms.

❸ $\sum_{x=5}^{8} (2x + 11)$

1. $2(5) + 11 = 21$
2. $2(6) + 11 = 23$
3. $2(7) + 11 = 25$

$2(8) + 11 = \underline{27}$

4. 96

❹ Write an arithmetic or geometric series in summation notation.

1. Find a formula for the n^{th} term (see ❷).
2. Replace "$A_n =$" with "Σ", write "$n = 1$" below it, and write the number of terms above it.

❹ Write the following series in summation notation.

a) the first 40 terms of 14 + 17 + 20 + 23 + …

1. $d = 3$

$A_n = 14 + 3(n - 1)$

$A_n = 3n + 11$

2. $\sum_{n=1}^{40} (3n + 11)$

b) -10 + 20 – 40 + 80 – 160 + 320

$r = -2$

$A_n = -10(-2)^{n-1}$

$A_n = 5(-2)^n$

$\sum_{n=1}^{6} 5(-2)^n$

❺ Count the number of terms in a series written in summation notation.

1. Subtract the number below Σ from the number above Σ.
2. Add 1 (to include the starting term).

❺ $\sum_{x=20}^{80} (3x - 16)$

1. 80 – 20 = 60
2. $n = 60 + 1 = 61$

The sum of the first n terms of an arithmetic series A is $S_n = \frac{n}{2}(A_1 + A_n)$.

The sum of the first n terms of a geometric series A is $S_n = A_1\left(\frac{1 - r^n}{1 - r}\right)$.

❻ Find the sum of an arithmetic or geometric series.

1. List the first few terms if needed.
2. Identify d or r and the first term. Do not round r. Label the first term A_1 (even if it is not term #1).
3. Identify the number of terms n.
4. If the series is arithmetic, identify the last term A_n.
5. Plug d or r, n, A_1, and A_n (if arithmetic) into the appropriate series formula above.

❻ Calculate the following sums.

a) the first 85 terms of -5 – 2 + 1 + 4 + …

1.
2. $d = 3$, $A_1 = -5$
3. $n = 85$
4. $A_{85} = -5 + 3(85 - 1) = 247$
5. $S_{85} = \frac{85}{2}(-5 + 247) = 10285$

b) $\sum_{x=8}^{100} (5x - 1)$

39 + 44 + 49 + 54 + …

$d = 5$, $A_1 = 39$

$n = 100 - 8 + 1 = 93$

$A_{100} = 499$

$S_{93} = \frac{93}{2}(39 + 499) = 25017$

c) $\sum_{x=1}^{30} 40\left(\frac{10}{9}\right)^x$

$r = \frac{10}{9}$, $A_1 = 40\left(\frac{10}{9}\right)^1 = \frac{400}{9}$

$n = 30 - 1 + 1 = 30$

$S_{30} = \frac{400}{9}\left(\frac{1 - (^{10}/_9)^{30}}{1 - {^{10}/_9}}\right) \approx 9036$

9-B Limits

The LIMIT of a Function f as x approaches a, written $\lim_{x\to a} f(x)$, is the value $f(a)$ would equal if it existed.

❶ Find the limit of a simple function at a given value.

1. If possible, plug in the value the independent variable approaches.
2. Otherwise, reason what value the function approaches as the independent variable approaches the value given.

❶

a) $\lim_{x\to 3} 2x$

$x = 3$ can be plugged in to the expression: $2(3) = 6$.

b) $\lim_{x\to 0} \frac{2}{|x|}$

$x = 0$ cannot be plugged in because $2 \div 0$ does not exist, but the closer x is to 0, the closer $2 \div |x|$ is to infinity, so the limit is infinity, that is, there is no limit.

For $|r| < 1$, $\lim_{n\to\infty} r^n = 0$. For example, multiplying ⅔ by itself over and over results in a smaller number each time, and the more times it is done, the closer the result is to zero. Therefore, if $|r| < 1$, r^n is zero in the formula for the sum or a geometric series with an infinite number of terms: $S_n = A_1\left(\frac{1-0}{1-r}\right)$, or $S_n = \frac{A_1}{1-r}$.

❷ Use an example and the concept of limits to derive the formula for the sum of an infinite geometric series.

1. Establish that $|r| < 1$.
2. Show that $\lim_{n\to\infty} r^n = 0$.
3. Substitute $r^n = 0$ in the formula for the sum of a geometric series (see 9-A).

❷ $120 + 60 + 30 + 15 + \ldots$

1. $r = ½$
2. $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$, because the more a number gets multiplied by ½, the closer it gets to zero.
3. $S_\infty = A_1\left(\frac{1-r^\infty}{1-r}\right) = 120\left(\frac{1-0}{1-½}\right) = \frac{120}{½} = 240$

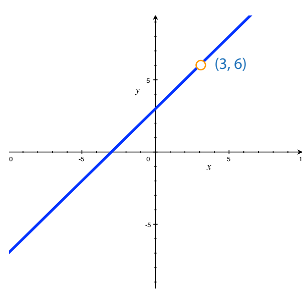
A function f is DISCONTINUOUS at a number a if there is a break in the graph at a. Though the value of $f(x)$ does not exist at a disconuity, its limit may exist.

❸ Use a graph to find the limit of a function f as it approaches a discontinuity.

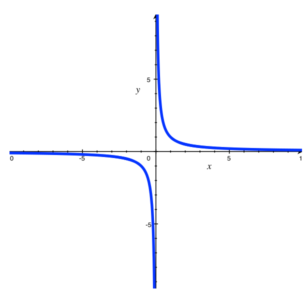
1. Graph f.
2. Find where the y value $f(a)$ would be on the graph if f were continuous, even if the actual value of $f(a)$ is undefined.
3. Infinity is not actually a number. Therefore, if a limit equals ∞, -∞, or both, the limit is said not to exist.

❸

a) $\lim_{x\to 3} \frac{x^2-9}{x-3} = 6$



b) $\lim_{x\to 0} \frac{1}{x} = \pm\infty$, so the limit does not exist



❹ Calculate values to estimate the limit of a function f as x approaches a number a.

1. Calculate $f(a)$. If it exists, then it is the limit. If it is undefined, continue with steps 2-6.
2. Calculate $f(a + .1)$.
3. Calculate $f(a + .01)$.
4. Calculate $f(a + .001)$.
5. If the calculated values are approaching ∞ or -∞, then no limit exists.

Otherwise, the limit is the value that the numbers calculated in steps 2-4 are approaching.

❹ $\lim_{x\to 3} \frac{x^2 - 9}{5x - 15}$

1. $f(3) = \frac{3^2 - 9}{15 - 15} = \frac{0}{0}$, which is undefined.
2. $f(3.1) = \frac{3.1^2 - 9}{15.5 - 15} = 1.22$
3. $f(3.01) = \frac{3.01^2 - 9}{15.05 - 15} = 1.202$
4. $f(3.001) = \frac{3.001^2 - 9}{15.005 - 15} = 1.2002$
5. The limit is 1.2.

A RATIONAL Function consists of one polynomial divided by another. Because this is undefined at each point where the denominator is zero, the function is discontinuous at these points. If a limit exists at such a point, it can be found by factoring the expression and canceling the factor that makes the denominator zero.

❺ Algebraically find the limit of a rational function f as x approaches a number a.

1. Factor the numerator and denominator.
2. Cancel factors.
3. Plug in $x = a$.

❺ $\lim_{x\to 3} \frac{x^2 - 9}{5x - 15}$

1. $\lim_{x\to 3} \frac{(x-3)(x+3)}{5(x-3)}$
2. $\lim_{x\to 3} \frac{x+3}{5}$
3. $\frac{3+3}{5} = 1.2$

9-C Slopes of Curves

The slope between two points is the change in y divided by the change in x. Using algebra, the slope formula can be written $m = \frac{y_2 - y_1}{x_2 - x_1}$. Using calculus, the slope of a function f at a given value of x can be designated $f'(x)$, and the slope formula can be written using function notation: $f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

The slope of a curve is not constant like the slope of a line. The slope of a curve at a specific point can be estimated by calculating the slope between that point and another point on the curve very close to it. The horizontal distance between the two points is $h = x_2 - x_1$.

❶ Calculate an estimate of the slope of a function at a given value of x.

1. Choose a value of h that is very close to zero, such as .001.
2. Calculate $f(x)$ and $f(x + h)$.
3. Calculate $f'(x) \approx \frac{f(x+h) - f(h)}{h}$.

❶ Estimate the slope of $f(x) = x^3$ when $x = 2$.

1. $h = .001$
2. $f(2) = 2^3 = 8$

   $f(2 + .001) = 2.001^3 \approx 8.012$
3. $f'(2) \approx \frac{8.012 - 8}{.001} = 12$

The closer h is to zero, the more accurate the estimated slope is. The exact slope is the limit of the slope function as h approaches zero: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

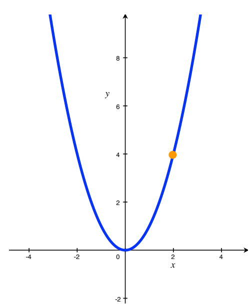
❷ Algebraically find the slope function of a power function, and use it to calculate the exact slope at a given value of x.

1. Expand $f(x + h)$ into a polynomial.
2. Substitute the polynomial form of $f(x + h)$ and of $f(x)$ into the slope formula.
3. Cancel $f(x)$ with the first term of $f(x + h)$.
4. Cancel h out of each term.
5. Plug in $h = 0$.
6. Plug the value of x into $f'(x)$.

❷ Find the slope of the given function when $x = 2$.

a) $f(x) = x^2$

1. $f(x + h) = (x + h)^2 = x^2 + 2xh + h^2$
2. $f'(x) = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h}$
3. $f'(x) = \lim_{h \to 0} \frac{2xh + h^2}{h}$
4. $f'(x) = 2x + h$
5. $f'(x) = 2x + 0$
6. $f'(2) = 2(2) = 4$



b) $f(x) = x^3$

$f(x + h) = (x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

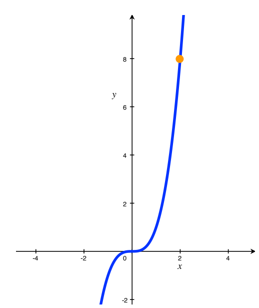
$f'(x) = \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^2}{h}$

$f'(x) = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$

$f'(x) = 3x^2 + 3xh + h^2$

$f'(x) = 3x^2 + 0 + 0$

$f'(2) = 3(2)^2 = 12$



Slopes can be added to other slopes and multiplied by coefficients. For example, given the slope of f is f' and the slope of g is g', the slope of $f + 2g$ is $f' + 2g'$.

❸ Calculate the exact slope of a polynomial function at a given value of x.

1. Find the expression for the slope of each individual power function in the polynomial (see ❷).
2. Add these expressions together to find the total slope function $f'(x)$.
3. Plug the value of x into $f'(x)$.

❸ Find the slope of $f(x) = x^3 + 10x^2$ when $x = 4$.

1. The slope of x^3 is $3x^2$, and the slope of x^2 is $2x$ (see ❷).
2. $f'(x) = 3x^2 + 10(2x) = 3x^2 + 20x$
3. $f'(4) = 3(4)^2 + 20(4) = 128$

9-D Derivatives

The DERIVATIVE of a Function $f(x)$ at a value a is the instantaenous rate of change of the function when $x = a$. This is the same as saying that the derivative of a function at a given point is the slope of the function at that point, as in 9-C: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}$. For example, if $f(t)$ is the temperature of a lake at time t, then $f'(t)$ is the rate at which the temperature of the lake is changing at time t.

The value of a derivative is given with respect to the independent variable. Using the lake example, if t is measured in seconds and $f(t)$ is measured in degrees, then $f'(t)$ is measured in degrees per second.

Some derivative functions have their own names, such as:

The derivative of position is *velocity*: Velocity is the rate at which position in a particular direction (which may be measured in meters) is changing with respect to time (which may be measured in seconds). For example, 20 meters per second is a measure of velocity.

The derivative of velocity is *acceleration*: Acceleration is the rate at which velocity (which may be measured in meters per second) is changing with respect to time (which may be measured in seconds). For example, 20 meters per second per second is a measure of acceleration.

The derivative of a function is its *slope*: Slope is the rate at which y is changing with respect to x, such as ⅓.

❶ State the deriviative function of a real-world function, using correct units.

1. The derivative function is the instantaneous rate of change of the independent variable with respect to the dependent variable.
2. The units of the derivative function are the units of the independent variable divided by ("per") the units of the dependent variable.

❶ $B(t)$ is the position of a ball, in meters from the start, after it has rolled for t seconds.

1. $B'(t)$ is the velocity of the ball exactly at a given time t. (Velocity means rate of change of position.)
2. The independent variable is measured in seconds and the dependent variable is measured in meters. Therefore, the velocity is measured in meters per second.

DIFFERENTIATION is the process of calculating a derivative. Most functions can be differentiated, such as those listed below.

function	derivative function	explanation
$f(x) = \sin x$	$f'(x) = \cos x$	The slope of a sine curve at any point is equal to the cosine of the x value at that point.
$f(x) = \cos x$	$f'(x) = -\sin x$	The slope of a cosine curve at any point is equal to the negative sine of the x value at that point.
$f(x) = e^x$	$f'(x) = e^x$	The slope of the exponential curve $y = e^x$ is equal to the value of x at that point.
$f(x) = \ln x$	$f'(x) = {}^1/_x$	The slope of the logarithmic curve $y = \ln x$ is equal to the reciprocal of x at that point.

Power functions have simple derivatives. The derivative functions for $f(x) = x^n$ are shown below for $n = 0$ through 4.

function	derivative function	explanation
$f(x) = c$	$f'(x) = 0$	The slope of a constant function (the horizontal line $y = c$) is 0 at all points.
$f(x) = x$	$f'(x) = 1$	The slope of the line $y = x$ is 1 at all points.
$f(x) = x^2$	$f'(x) = 2x$	The slope of the parabola $y = x^2$ at any point is equal to double the x value at that point.
$f(x) = x^3$	$f'(x) = 3x^2$	The slope of $y = x^3$ at any point is equal to triple the square of the x value at that point.
$f(x) = x^4$	$f'(x) = 4x^3$	The slope of $y = x^4$ at any point is equal to four times the cube of the x value at that point.

These derivative functions can be generalized by the POWER Rule: The derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$. Combining this with the fact that derivative functions can be added together and multiplied by constants makes it easy to find the derivative of polynomial functions.

❷ Find the derivative of a polynomial function.

1. For each term ax^n, multiply the coefficient a by the exponent n and then subtract 1 from n. Keep in mind that a linear term such as $5x$ has an exponent of 1 and a constant term such as 5 has an exponent of 0.

❷ Find the derivative function for $f(x) = 6x^{10} + x^3 - 4x^2 + 7x - 9$.

1. $f'(x) = 10(6)x^{10-1} + 3x^{3-1} - 2(4)x^{2-1} + 1(7)x^{1-1} - 0(9)x^{0-1} = 60x^9 + 3x^2 - 8x + 7 - 0$

❸ Use derivatives to find the slope of a function at a given point.

1. Find the derivative function.
2. Plug the given value of x into the derivative function.

❸ Find the slope of the following curves at the given points.

a) $f(x) = 5x^4 - 2x$, at $x = 3$

1. $f'(x) = 20x^3 - 2$
2. $f'(3) = 20(3)^3 - 2 = 538$

b) $g(x) = \sin x$, at $x = 60°$

$g'(x) = \cos x$

$g'(60°) = \cos 60° = ½$

A Line that is TANGENT to a Curve is one that touches the curve at one point but does not cross it. The slope of a tangent line is the same as the slope of the curve at that point.

❹ Find the equation of a line tangent to a curve at a given x-value a.

1. Plug a into $f(x)$ to find the function's value $f(a)$ at a.
2. Find the derivative function $f'(x)$.
3. Plug a into $f'(x)$ to find the function's slope $f'(a)$ at a.
4. Plug in $x = a$, $y = f(a)$, and $m = f'(a)$ to the equation of a line $y = mx + b$.
5. Solve for b.
6. The equation of the tangent line is $y = mx + b$ where x and y are variables, $m = f'(a)$ is the slope from step 3, and b is the y-intercept from step 5.

❹ Find the equation of the line tangent to $f(x) = x^2 - 8x + 5$ at $x = 2$.

1. $f(2) = 2^2 - 8(2) + 5 = -7$
2. $f'(x) = 2x - 8$
3. $f'(2) = 2(2) - 8 = -4$
4. $-7 = -4(2) + b$
5. $1 = b$
6. $y = -4x + 1$