Chapter Eight

Vectors

Two-Dimensional Vectors

Vector Equations

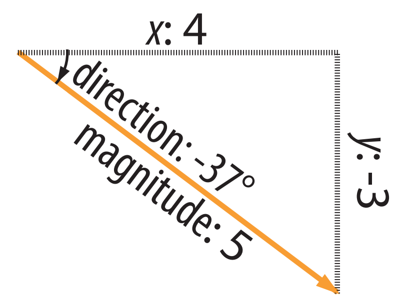
Three-Dimensional Vectors

Unit Vectors

Angles Between Vectors

Two-dimensional vectors

A **vector** is a magnitude and a direction. A vector is symbolized by a bold letter, such as **v**. When handwritten, a vector is symbolized by a letter with a right-facing arrow above it, such as $\vec{v}$.



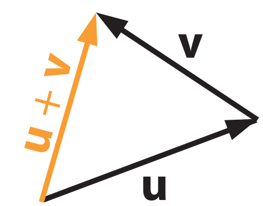
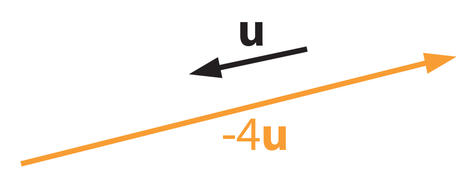
A vector can be represented graphically by an arrow, or numerically by its components. When represented by its components, it is commony written $<x, y>$, $\begin{bmatrix} x \\ y \end{bmatrix}$, or $\begin{pmatrix} x \\ y \end{pmatrix}$.

Concept	Definition	How to find	Example: v = <4, -3>
x-component	total distance the vector extends horizontally	given	$x = 4$
y-component	total distance the vector extends vertically	given	$y = -3$
magnitude	"length" of the vector	$\lVert\mathbf{v}\rVert = \sqrt{x^2 + y^2}$	$\lVert\mathbf{v}\rVert = \sqrt{4^2 + (-3)^2} = 5$
direction	angle made with the positive *x*-axis	$\theta = \tan^{-1} \frac{y}{x}$ (add 180° if *x* is negative)	$\theta = \tan^{-1} \frac{-3}{4} = -37°$

Basic vector arithmetic

Vectors can be added to other vectors.

Vectors can be multiplied by a number, called a **scalar**, to change their magnitude. If a scalar is negative, it will reverse the direction of the vector.

Operation	Algebraic	Geometric
Addition	Add the *x* components together, and add the *y* components together. $\begin{pmatrix}8\\3\end{pmatrix}+\begin{pmatrix}-6\\4\end{pmatrix}=\begin{pmatrix}2\\7\end{pmatrix}$	Sketch the second vector starting where the first one ends. Connect the start of the first to the end of the second. 
Scalar Multiplication	Multiply each component by the scalar. $-4\begin{pmatrix}8\\3\end{pmatrix}=\begin{pmatrix}-32\\-12\end{pmatrix}$	Make the vector *k* times as long, where *k* is the scalar. If *k* is negative, reverse its direction. 

Vector equations

Although vectors themselves do not have position, a **position vector** shows the position of a point with respect the origin. The position vector $<x, y>$ represents the point (x, y).

With this definition, a line can be determined by a vector equation.

Aspect	Regular	Vector
Equation	$y = mx + b$	$\mathbf{r} = <a, b> + t<x, y>$
Slope	m	${}^{y}/_{x}$
Known Point	$(0, b)$	(a, b)
Independent Variable	x	t
Dependent Variable	y	$\mathbf{r}$
Does the line pass through the point (c, d)?	Plug in $x = c$. If this makes if y equal to d, the point is on the line.	Solve for t in the equations $a + tx = c$ and $b + ty = d$. If t is the same in both equations, the point is on the line.
Where do two lines intersect?	Solve for x in the equation $m_1x + b_1 = m_2x + b_2$ and plug it in to find y.	Solve for t_1 or t_2 in the system $\begin{cases} a_1 + t_1x_1 = a_2 + t_2x_2 \\ b_1 + t_1y_1 = b_2 + t_2y_2 \end{cases}$ and plug it in.

Three-dimensional vectors

Adding in a *z* component makes a vector three-dimensional.

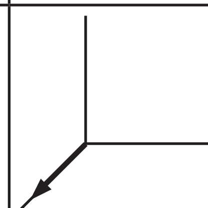
When applicable, methods used for two-dimensional vectors apply to three-dimensional vectors as well, such as in the examples below.

Concept	Two-dimensional example	Three-dimensional example
Vector from a point to a point	The vector from (4, 10) to (5, 7) is $\begin{pmatrix} 5-4 \\ 7-10 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$	The vector from (4, 10, 3) to (5, 7, 9) is $\begin{pmatrix} 5-4 \\ 7-10 \\ 9-3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 6 \end{pmatrix}$
Magnitude	The magnitude of <4, 8> is $\sqrt{4^2+8^2} = \sqrt{80} \approx 8.94$	The magnitude of <4, 8, 3> is $\sqrt{4^2+8^2+3^2} = \sqrt{89} \approx 9.43$
Addition	$2\begin{pmatrix} 4 \\ 8 \end{pmatrix} + 3\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$	$2\begin{pmatrix} 4 \\ 8 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 6 \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ 3 \end{pmatrix}$

Unit Vectors

A **unit vector** is a vector with a magnitude of 1.

Three specific unit vectors have their own names.

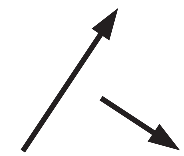
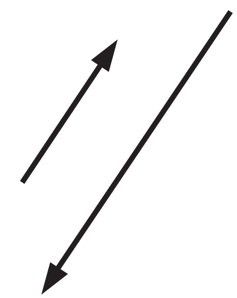
Unit Vector	Component Form	Sketch
i	<1, 0, 0>	
j	<0, 1, 0>	
k	<0, 0, 1>	

Any vector can be written as the sum of multiples of **i**, **j**, and **k**. For example, $\langle 3, -2, 1\rangle = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

A vector divided by its magnitude is a unit vector in the same direction. For example, if $\mathbf{v} = \langle 3, 4\rangle$, then $\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = 5$, and $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \langle \frac{3}{5}, \frac{4}{5}\rangle$ is a unit vector.

Angles between vectors

The **dot product** of two vectors is the sum of the products of the corresponding components. For example, $\langle 3, 10\rangle \cdot \langle -2, 5\rangle = 3(-2) + 10(5) = -6 + 50 = 44$.

Relationship	How to find	Example
Orthogonal	If the dot product is zero, the angle between the vectors is 90°.	$\mathbf{u} = \langle 4, 6\rangle$ $\mathbf{v} = \langle 3, -2\rangle$  $\mathbf{u} \cdot \mathbf{v} = 4(3) + 6(-2) = 12 - 12 = 0$
Parallel	If one vector is a scalar multiple of the other, the angle between them is 0° if the scalar is positive or 180° if it is negative.	$\mathbf{u} = \langle 4, 6\rangle$ $\mathbf{v} = \langle -8, -12\rangle$ $\mathbf{v} = -2\mathbf{u}$ 
Neither	$\theta = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{\lVert\mathbf{u}\rVert\,\lVert\mathbf{v}\rVert}$	$\mathbf{u} = \langle 4, 6\rangle$ $\mathbf{v} = \langle 5, 2\rangle$ $\theta = \cos^{-1} \frac{4(5) + 6(2)}{\sqrt{(4^2 + 6^2)(5^2 + 2^2)}}$ $= \cos^{-1} \frac{32}{\sqrt{1508}}$ $\approx 34.5°$