CHAPTER EIGHT: VECTORS

Review April 25 ➳ Test May 2

A vector is a line segment with a specified length and direction but no specified position. Vectors have an x component, a y component, and, if three-dimensional, a z-component. For example, the vector <3, -4> has an x component of 3 and a y component of -4, making for a slope of $^{y}/_{x} = {}^{-4}/_{3}$ and a length of $\sqrt{(-4)^2 + 3^2} = 5$. Vectors are used extensively in calculus and physics applications, although only basics are covered in this chapter.

8-A Two-Dimensional Vectors — Monday • 4/10

vector • tail • head • component • magnitude • scalar

1. Identify the vector from one point to another.
2. Find the magnitude of a vector.
3. Multiply a vector by a scalar.
4. Find sums and differences of vectors algebraically.
5. Find sums and differences of vectors geometrically.
6. Find the angle a vector makes with a horizontal line.

8-B Vector Equations — Thursday • 4/13

position vector

1. Write a vector equation of a line through a given point and parallel to a given vector.
2. Write a vector equation of a line through two points.
3. Determine whether or not a line passes through a given point.
4. Find the point of intersection of two lines written as vector equations.

8-C Three-Dimensional Vectors — Tuesday • 4/18

1. Apply techniques for two-dimensional vectors to three-dimensional vectors.

8-D Unit Vectors — Thursday • 4/20

unit vector • **i** • **j** • **k**

1. Express a vector as a linear combination of unit vectors.
2. Find a unit vector parallel to a given vector.

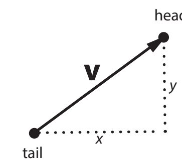
8-E Angles Between Vectors — Tuesday • 4/25

dot product • orthogonal

1. Find the dot product of two vectors.
2. Find the angle between two vectors.
3. Determine whether two vectors are orthogonal, parallel, or neither.

8-A Two-Dimensional Vectors

A directed line segment is a line segment that has a starting point (a TAIL) at one end and an ending point (a HEAD) at the other end.



A VECTOR is the set of all directed line segments in a specified direction and with a specified length, such as "two units upward". Any such directed line segment can be drawn to represent the vector.

Vectors do not have position. This is different from lines, parabolas, etc., which can be translated to a different position but become a different line, parabola, etc.—with a different equation—in the process. Drawing a vector in a different place does not change the vector itself.

The Horizontal COMPONENT of a Vector is the horizontal distance from its tail to its head. Likewise, the Vertical Component is the vertical distance. A vector expressed as its components is in Component Form, usually written <*x*, *y*>, $\begin{bmatrix} x \\ y \end{bmatrix}$, or $\begin{pmatrix} x \\ y \end{pmatrix}$.

In print, vectors are represented by bold lowercase letters, such as **v**. When handwritten, a right-facing arrow is placed above the letter, such as $\vec{v}$.

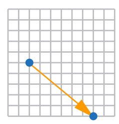
❶ Identify the vector from one point to another.

1. Subtract the *x* component of the first vector from the *x* component of the second vector.
2. Repeat step 1 with the *y* components.

❶ Sketch the following vectors, and express them in component form.

a) **a** goes from (2, 5) to (8, 0)
1. 8 – 2 = 6
2. 0 – 5 = -5

<6, -5>

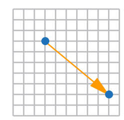


b) **b** goes from (3, 7) to (9, 2)

9 – 3 = 6

2 – 7 = -5

<6, -5>

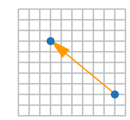


c) **c** goes from (9, 2) to (3, 7)

3 – 9 = -6

7 – 2 = 5

<-6, 5>



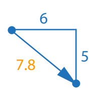
The length of a vector is called its MAGNITUDE. The magnitude of **a** is written ||**a**|| or |**a**|, and can be found by the Pythagorean theorem: $||\mathbf{a}|| = \sqrt{x^2 + y^2}$.

❷ Find the magnitude of a vector.

1. Use the Pythagorean theorem.

❷ **h** = <6, -5>

1. $||\mathbf{h}|| = \sqrt{6^2 + (-5)^2} = \sqrt{61} \approx 7.8$



A SCALAR is a number multiplied by a vector to change its magnitude. If the scalar is negative, it will also reverse the vector's direction.

❸ Multiply a vector by a scalar.

1. Multiply each component by the scalar.

❸ Find -10**a** given **a** = <2, -3>.

1. <-10(2), -10(-3)> = <-20, 30>

Vectors can be added or subtracted.

❹ Find sums and differences of vectors algebraically.

1. Multiply in any scalars.
2. Add or subtract the corresponding components of the vectors.

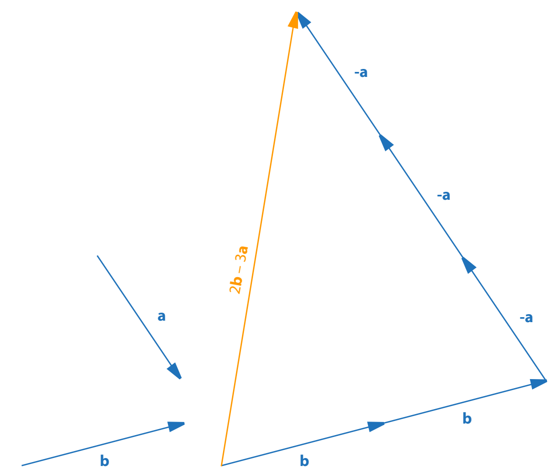
❹ Find 2**a** – 10**b** given **a** = <8, 6> and **b** = <2, -1>.

1. <16, 12>
– <20, -10>
2. < -4, 22>

❺ Find sums and differences of vectors geometrically.

1. Sketch the first vector.
2. Sketch the second vector, positioning its tail at the head of the first vector.
3. Repeat step 2 for each additional vector.
4. The resulting vector goes from the start of the first vector to the end of the last vector.

❺ Find 2**b** – 3**a** given **a** and **b** shown at right.



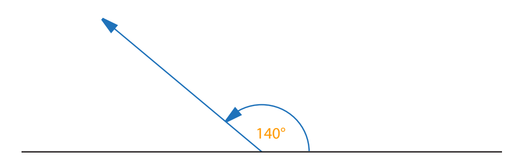
The slope of a vector <x, y> is $m = \frac{y}{x}$. Therefore, the angle a vector <x, y> makes with the x-axis is $\theta = \tan^{-1}\frac{y}{x}$.

❻ Find the direction of a vector.

1. Divide the y component by the x component to get the slope.
2. Take the tangent inverse of the slope.
3. If the x component is negative, add 180°.

❻ **a** = <-6, 5>

1. The slope is $\frac{5}{-6}$.
2. $\tan^{-1}\left(\frac{5}{-6}\right) \approx -40°$
3. -40° + 180° = 140°



8-B Vector Equations

A POSITION Vector of a Point P is the vector that defines where P is with respect to the origin. The position vector for the point (x, y) is $<x, y>$. The equation $\mathbf{r} = t<x, y> + <a, b>$ is a line with slope $^{y}/_{x}$. $\mathbf{r}$ is the set of points defined by the position vectors $t<x, y> + <a, b>$.

❶ Write a vector equation of a line through a given point and parallel to a given vector.

1. In the equation above, use x and y to represent the slope and use a and b to represent the point.

❶ Write the equation of the line through the point (3, 4) in the direction $<5, -2>$.

1. $\mathbf{r} = t<5, -2> + <3, 4>$

❷ Write a vector equation of a line through two points.

1. Find the vector (in either direction) between the two points (see ❶ in 8-A).
2. Do ❶ above, using the vector from step 1 and either given point.

❷ Write the equation through the points (9, 5) and (-3, 1).

1. $\mathbf{a} = <9 - -3, 5 - 1> = <12, 4>$
2. $\mathbf{r} = t<12, 4> + <9, 5>$

❸ Determine whether or not a line passes through a given point.

1. Set the equation equal to a position vector representing the point.
2. Solve for the scalar variable using only the x components.
3. Repeat step 2 with the y components.
4. If the result is the same each time, then the line passes through the point.

❸ Does the line $\mathbf{r} = t<10, 4> + <-9, 5>$ pass through the point (21, 17)?

1. $t<10, 4> + <-9, 5> = <21, 17>$
2. $10t - 9 = 21$ $\qquad t = 3$
3. $4t + 5 = 17$ $\qquad t = 3$
4. 3 = 3, so the line does pass through the point (21, 17).

❹ Find the point of intersection of two lines written as vector equations.

1. Write an equation setting the x component of the first line equal to the x component of the second line.
2. Repeat step 1 for the y component.
3. Solve the system of equations from steps 1 and 2 for either variable.
4. Plug the solution from step 3 into the corresponding line equation. The resulting vector represents the point of intersection.

❹ Find the point of intersection of the lines $\mathbf{r} = t<2, 3> + <5, -10>$ and $\mathbf{q} = s<8, -4> + <-7, 12>$.

1. $2t + 5 = 8s - 7$
2. $3t - 10 = -4s + 12$
3. $2t - 8s = -12 \qquad 2t - 8s = -12$

   $3t + 4s = 22 \qquad \underline{6t + 8s = 44}$

   $\qquad\qquad\qquad 8t \quad = 32 \qquad t = 4$
4. $4<2, 3> + <5, -10> = <13, 2>$, so the point is (13, 2)

8-C Three-Dimensional Vectors

When applicable, methods for working with three-dimensional vectors are the same as those for two-dimensional vectors.

❶ Apply techniques for two-dimensional vectors to three-dimensional vectors.

1. Follow the directions for two-dimensional vectors (see 8-A or 8-B), but add in a *z* component.

❶ Find the magnitude of **a** = <5, -4, 2>.

1. $||\mathbf{a}|| = \sqrt{5^2 + (-4)^2 + 2^2} = \sqrt{45} \approx 6.7$

❶ Is the point (12, 9, 4) on the line **a** = *t*<4, 6, -2> + <0, -9, 12>?

1. *t*<4, 6, -2> + <0, -9, 12> = <12, 9, 4>

$4t + 0 = 12$ $t = 3$

$6t - 9 = 9$ $t = 3$

$-2t + 12 = 4$ $t = -4$

The point is not on the line because the scalar *t* is not the same for all three components.

8-D Unit Vectors

A UNIT Vector is a vector with a magnitude of 1.

The unit vector in the positive x direction, <1, 0, 0>, is called **i**.

The unit vector in the positive y direction, <0, 1, 0>, is called **j**.

The unit vector in the positive z direction, <0, 0, 1>, is called **k**.

Any vector can be written as a linear combination of **i**, **j**, and **k**: $\langle x, y, z\rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

❶ Express a vector as a linear combination of unit vectors.

1. Write the vector as the sum of each component times the unit vector in that direction (e.g., **i** for the x direction).

❶ <10, 5, -3>

$10\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$

Dividing a vector by its magnitude yields a unit vector in the same direction: $\mathbf{u} = \frac{\mathbf{a}}{||\mathbf{a}||}$.

❷ Find a unit vector parallel to a given vector.

1. Find the magnitude of the vector.
2. Divide the vector by its magnitude.

❷ $\mathbf{a} = 10\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$

$||\mathbf{a}|| = \sqrt{10^2 + 5^2 + (-3)^2} = \sqrt{134} \approx 11.6$

$\frac{\mathbf{a}}{11.6} = \frac{10}{11.6}\mathbf{i} + \frac{5}{11.6}\mathbf{j} - \frac{3}{11.6}\mathbf{k} \approx .86\mathbf{i} + .43\mathbf{j} - .26\mathbf{k}$ (that is, <.86, .43, -.26>)

8-E Angles Between Vectors

The DOT Product of two vectors is the sum of the products of their corresponding components: $<x_1, y_1, z_1> \bullet <x_2, y_2, z_2> = x_1(x_2) + y_1(y_2) + z_1(z_2)$.

❶ Find the dot product of two vectors.

1. Multiply each component of the first vector by the corresponding component of the second vector.
2. Add these products together.

❶ Find $\mathbf{a} \bullet \mathbf{b}$ given $\mathbf{a} = <10, 2, 5>$ and $\mathbf{b} = <-3, 7, 1>$.

$\mathbf{a} \bullet \mathbf{b} = 10(-3) + 2(7) + 5(1) = -11$

The angle between vectors **a** and **b** is $\theta = \cos^{-1} \frac{\mathbf{a} \bullet \mathbf{b}}{||\mathbf{a}||\,||\mathbf{b}||}$.

❷ Find the angle between two vectors.

1. Find the dot product of the vectors.
2. Find the magnitude of each vector.
3. Multiply the magnitudes together.
4. Divide the dot product in step 1 by the product of magnitudes in step 3.
5. Take the inverse cosine of the quotient in step 4.

❷ Find the angle between $\mathbf{a} = <4, 5, -10>$ and $\mathbf{b} = <8, 0, -1>$.

1. $\mathbf{a} \bullet \mathbf{b} = 4(8) + 5(0) - 10(-1) = 42$
2. $||\mathbf{a}|| = \sqrt{4^2 + 5^2 + (-10)^2} = \sqrt{141} \approx 11.9$
   $||\mathbf{b}|| = \sqrt{8^2 + 0^2 + (-1)^2} = \sqrt{65} \approx 8.1$
3. $||\mathbf{a}||\,||\mathbf{b}|| \approx 11.9 \bullet 8.1 \approx 96.4$
4. $\frac{\mathbf{a} \bullet \mathbf{b}}{||\mathbf{a}||\,||\mathbf{b}||} \approx \frac{42}{96.4} \approx .436$
5. $\theta \approx \cos^{-1} .436 \approx 64.2°$

In vector geometry, ORTHOGONAL means perpendicular.

Two vectors **a** and **b** are orthogonal if $\frac{\mathbf{a} \bullet \mathbf{b}}{||\mathbf{a}||\,||\mathbf{b}||}$ equals 0, because $\cos^{-1} 0 = 90°$. This happens if their dot product is zero.

Two vectors **a** and **b** are parallel if $\frac{\mathbf{a} \bullet \mathbf{b}}{||\mathbf{a}||\,||\mathbf{b}||}$ equals 1 or -1, because $\cos^{-1} 1 = 0°$ and $\cos^{-1} -1 = 180°$. This happens if $\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$.

❸ Determine whether two vectors are orthogonal, parallel, or neither.

1. Find the dot product of the vectors. If it is zero, the vectors are orthogonal.
2. If the vectors are not orthogonal, divide each component in the first vector by the corresponding component in the second vector. If the quotients are all equal, the vectors are parallel.

❸ $\mathbf{a} = <-3, -20, 12>$ and $\mathbf{b} = <-2, 5, 8>$

1. $\mathbf{a} \bullet \mathbf{b} = -3(-2) + -20(5) + 12(8) = 2 \neq 0$, so the vectors are not orthogonal.
2. $\frac{x_1}{x_2} = \frac{-3}{-2} = 1.5$, $\frac{y_1}{y_2} = \frac{-20}{5} = -4$, $\frac{z_1}{z_2} = \frac{12}{8} = 1.5$. $-4 \neq 1.5$, so the vectors are not parallel.