Chapter Six

Probability

Counting Methods

Set Notation and Venn Diagrams

Probability of a Single Event

Probability of Specific Multiple Events

Probability of General Multiple Events

Probability Distributions

Combinations

A **combination** is a group of selected items.

The number of possible combinations of r items in a group of n items is ***n* choose *r***, written $\binom{n}{r}$ or ${}_nC_r$.

Group of *n* elements	Chosen group of *r* elements	Combination Example	Number of possible combinations
7 days of the week	1 day	Thursday	$\binom{7}{1} = 7$
12 months of the year	4 months	June, September, October, December	$\binom{12}{4} = 495$

The **counting principle** states that if two independent events have a and b possible outcomes, respectively, then there are a total of ab possible outcomes for the two events. This can be expanded to abc possible outcomes for three events, etc.

Events	Outcome Example	Number of possible outcomes
1 day of the week and 1 month of the year	Thursday; October	$\binom{7}{1}\binom{12}{1} = 84$
1 day of the week and 4 months of the year	Thursday; June, September, October, December	$\binom{7}{1}\binom{12}{4} = 3465$

Choosing one element at a time

Events	Outcome Example	Number of possible outcomes
Roll 3 six-sided dice	6; 3; 5	$\binom{6}{1}\binom{6}{1}\binom{6}{1} = 216$
Choose 3 favorite colors, in order, out of 8 colors	black; red; orange	$\binom{8}{1}\binom{7}{1}\binom{6}{1} = 336$
Choose a background color, a border color, and a text color, out of 8 colors	red; black; white	$\binom{8}{1}\binom{7}{1}\binom{6}{1} = 336$

For independent events (values can be repeated) like the dice example, the calculation can be simplified to the exponential n^r, in this case 6^3.

For dependent events (values cannot be repeated) like the colors examples, the calculation can be simplified to the permutation ${}_nP_r$, in this case ${}_8P_3$. A **permutation** is a combination in which each item selected is assigned a specific value, such as *first*, *second*, and *third*, or *background*, *border*, and *text*.

Set Notation

A **set** is a combination.

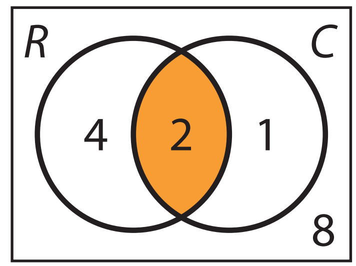
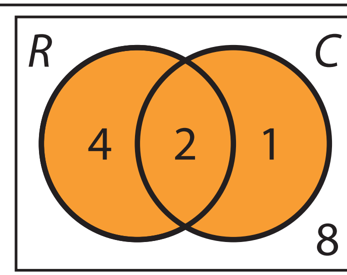
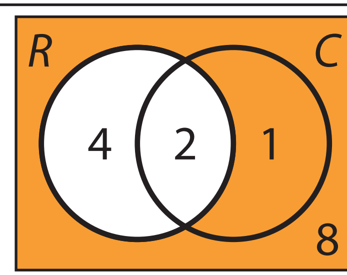
The examples below use the sets $A = \{\text{Saturday, Sunday}\}$ and $B = \{\text{Sunday, Tuesday, Thursday}\}$.

Term	Definition	Notation and Example
Element of *A*	item in *A*	Saturday $\in A$
Cardinality of *A*	number of elements in *A*	$\|A\| = 2$
Intersection of *A* and *B*	set of elements in both *A* and *B*	$A \cap B = \{\text{Sunday}\}$
Union of *A* and *B*	set of elements in either *A* or *B*	$A \cup B = \{\text{Sunday, Tuesday, Thursday, Saturday}\}$
Complement of *A*	set of elements not in *A*	$A' =$ the set of weekdays
Universal Set	set of all elements in the given context	$U =$ the set of days of the week
Empty Set	set containing no elements	$\emptyset = \{\ \}$

Venn Diagrams

A **Venn diagram** is used to represent the relationship between two or three sets, each of which is represented by a circle.

The examples below represent a bag of 6 red marbles (set R) and 9 yellow marbles, including 2 red marbles and 1 yellow marble that are cracked (set C).

Set notation	Meaning	Venn Diagram	Number of marbles
$R \cap C$	red and cracked		$\lvert R \cap C \rvert = 2$
$R \cup C$	red or cracked		$\lvert R \cup C \rvert = 7$
R'	not red		$\lvert R' \rvert = 9$

The probability of an event

The **sample space** of an event is the set of all possible outcomes.

The probability of an event A, written $P(A)$, can be defined as the number of outcomes satsifying A divided by the total number of outcomes in the sample space: $P(A) = \frac{|A|}{|U|}$. For this definition to apply, the outcomes must all be equally likely.

Event A	Outcomes satisfying A	Sample space	Probability of A
Roll higher than 2 on a 6-sided die	3, 4, 5, 6	1, 2, 3, 4, 5, 6	$P(A) = \frac{2}{6}$
Out of three coin flips, two are heads and one is tails.	HHT, HTH, THH	HHH, HHT, HTH, THH, HTT, THT, TTH, TTT	$P(A) = \frac{3}{8}$

It is not incorrect to reduce probabilities or convert them to decimals or percents, but doing so removes information about the event. Do not do so in this course, except for fractions equaling 0 or 1.

Given information

Probability is based on the information known, regardless of what has happened. **Conditional** probability takes into account **given** (known) conditions.

Event	Probability	Given condition	Conditional probability
A card is hearts.	$\frac{13}{52}$	The card is red.	$\frac{13}{26}$
The second card drawn is hearts.	$\frac{13}{52}$	The first card was hearts.	$\frac{12}{51}$
The third card drawn is hearts.	$\frac{13}{52}$	The next 12 cards are all hearts.	$\frac{1}{40}$

Probabilities of multiple events

For probability problems involving multiple events, the individual probabilities can be multiplied together. Keep in mind that, in some cases, the individual probabilities change based on the events already accounted for, such as fewer cards being left in a deck as more cards are drawn. Such events are called **dependent events**.

Events	Type	Probability	Comment
Roll 3 dice; all land on 6.	independent	$(\frac{1}{6})(\frac{1}{6})(\frac{1}{6}) = \frac{1}{216}$	Each die roll is unaffected by the others.
Draw 3 cards; all are aces.	dependent	$(\frac{4}{52})(\frac{3}{51})(\frac{2}{50}) = \frac{24}{132600}$	Once you know an ace has been drawn, there are only 3 aces possible to draw out of the remaining 51 cards.
Draw 3 cards, putting them back each time; all are aces.	independent	$(\frac{4}{52})(\frac{4}{52})(\frac{4}{52}) = \frac{64}{140608}$	There are always 4 aces available. You could draw the same ace each time.
Draw 3 cards; they are hearts, clubs, king, in that order.	dependent (hearts and clubs), independent (king)	$(\frac{13}{52})(\frac{13}{51})(\frac{4}{52}) = \frac{676}{137904}$	Even though you know you drew a heart and a club, you have no information about whether or not you drew a king. All 52 cards are still equally likely to be a king.

Different arrangements of multiple events

Some events can happen in different orders or ways. For example, rolling a 2 and a 5 on two dice could be (2, 5) or (5, 2). To find the probability of such an event, every possible order or way must be included.

In many cases, each possibility has the same probability, so can be calculated just once and then multiplied by the number of possibilities there are. This number can be found using choose.

The examples below refer to pulling three marbles out of a jar of 5 orange, 3 red, and 1 black marble.

Event	Possibilities	Probability
All three marbles are the same color.	All blue: $P(\text{blue, blue, blue}) = (\frac{5}{9})(\frac{4}{8})(\frac{3}{7})$ All red: $P(\text{red, red, red}) = (\frac{3}{9})(\frac{2}{8})(\frac{1}{7})$	$(\frac{5}{9})(\frac{4}{8})(\frac{3}{7}) + (\frac{3}{9})(\frac{2}{8})(\frac{1}{7}) = \frac{66}{504}$
Two marbles are red and one is blue.	$P(\text{red, red, blue}) = (\frac{3}{9})(\frac{2}{8})(\frac{5}{7})$ $P(\text{red, blue, red}) = (\frac{3}{9})(\frac{5}{8})(\frac{2}{7})$ $P(\text{blue, red, red}) = (\frac{5}{9})(\frac{2}{8})(\frac{2}{7})$ There are $\binom{3}{2}$ ways to choose which two of the three marbles are red. Each way has the same probability.	$\binom{3}{2}(\frac{3}{9})(\frac{2}{8})(\frac{5}{7}) = \frac{90}{504}$
All three marbles are different colors.	There are $\binom{3}{1}\binom{2}{1}\binom{1}{1}$ ways to choose which marble is blue, which is red, and which is black. Each way has the same probability.	$\binom{3}{1}\binom{2}{1}\binom{1}{1}(\frac{5}{9})(\frac{3}{8})(\frac{1}{7}) = \frac{90}{504}$

Using the complement

In many cases, the probability of an event's complement is easier to calculate than the probability of the event itself. Since the complement is everything the original event is not, these two probabilities must add up to 100%: $P(A) = 1 - P(A')$. For example, if there is an 80% chance it will rain today, there is a 20% chance it will not rain today.

The examples below refer to rolling four 6-sided dice.

Event A	Complement A'	$P(A')$	$P(A)$
Roll at least one 6.	Roll zero 6's.	$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right) = \frac{625}{1296}$	$1 - \frac{625}{1296} = \frac{671}{1296}$
Roll at least two 6's.	Roll zero or one 6.	$\frac{625}{1296} + \binom{4}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right) = \frac{1125}{1296}$	$1 - \frac{1125}{1296} = \frac{171}{1296}$

Binomial Probabilities

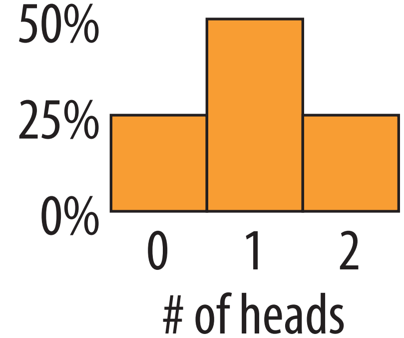
A **binomial experiment** is a scenario in which a specific independent event is attempted multiple times so see how many successes there are.

Value	Meaning	Example: 3 correct predictions in ten 6-sided die rolls
n	number of trials	10 rolls were made.
r	number of successes	3 rolls were correctly predicted.
p	probability of success on each individual trial	Each roll had a $\frac{1}{6}$ chance of being correctly predicted.
q	probability of failure on each individual trial ($q = p'$)	Each roll had a $\frac{5}{6}$ chance of being incorrectly predicted.
p^r	probability of r successes out of r trials	There is a $(\frac{1}{6})^3 = \frac{1}{216}$ chance of three rolls in a row being correctly predicted.
q^{n-r}	probability of $n - r$ failures out of n trials	There is a $(\frac{5}{6})^7 = \frac{78125}{279936}$ chance of seven rolls in a row being incorrectly predicted.
$\binom{n}{r}$	number of possible orders of r successes out of n total trials	There are $\binom{10}{3} = 120$ different ways to choose which 3 of the 10 rolls were correctly predicted.
$\binom{n}{r}p^r q^{n-r}$	probability of exactly r successes (and $n - r$ failures) out of n trials	The probability of correctly predicting exactly 3 out of 10 rolls on a 6-sided die is $\binom{10}{3}(\frac{1}{6})^3(\frac{5}{6})^7 = 120(\frac{1}{216})(\frac{78125}{279936}) = \frac{9375000}{60466176} \approx 15.5\%$

Probability Distributions

A **probability distribution** states each possible outcome or range of outcomes of an event and how likely it is.

A probability distribution can be displayed in many ways, such as a sentence, table, or graph. When the variable is numerical, a histogram is commonly used.

Flip two coins.	Sentence	Table	Graph
Probability distribution for number of heads	There is a 25% chance of getting either 0 or 2 heads, and a 50% chance of getting exactly 1 head.	Probability of exactly r heads out of 2 coin flips: r | $P(r)$ 0 | 25% 1 | 50% 2 | 25%	Probability of exactly r heads out of 2 coin flips 

Expected Value

The **expected value** μ of a probability distribution is the expected average of infinitely many random values of the distribution. For example, if you win \$10 for tails and \$30 for heads, your expected value is \$20.

Expected value can be calculated by multiplying each possible value of the distribution by its probability and adding these products. The example below calculates the expected number of 4's out of 2 rolls on a 4-sided die.

Event x	Value	Probability $P(x)$	Product $xP(x)$
no 4's	0	$\binom{2}{0}\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)=\frac{9}{16}$	$0\left(\frac{9}{16}\right)=0$
one 4	1	$\binom{2}{1}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)=\frac{6}{16}$	$1\left(\frac{6}{16}\right)=0.375$
two 4's	2	$\binom{2}{2}\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)=\frac{1}{16}$	$2\left(\frac{1}{16}\right)=0.125$
Total:			$\mu=0+0.375+0.125=\mathbf{0.5}$

For binomial distributions, expected value can be calculated simply as $\mu = np$. In the example above, $\mu = 2\left(\frac{1}{4}\right) = \mathbf{0.5}$.