CHAPTER SIX: PROBABILITY

Review March 2 ↬ Test March 16

The probability of an event is the number of ways it can successfully happen divided by the total possible number of ways. To maintain this definition, probability problems in this chapter are done with unreduced fractions. Many people find probability to be difficult because it involves reasoning and can be counterintuitive. For example, people commonly do not understand that probability is based on what information is known, regardless of what events have actually occurred. Despite its difficulty, some people like probability because it is more readily applicable than other topics.

6-A Counting Methods — Thursday • 2/9

combination • choose • sample space • fundamental counting principle • permutation

1. Count combinations.
2. Find the total number of possible outcomes in a series of events.
3. Count permutations.
4. Use a graphing calculator to count combinations and permutations.
5. Find the size of a sample space by using permutations, if possible.

6-B Set Notation and Venn Diagrams — Tuesday • 2/14

set • element • subset • cardinality • intersection • union • complement • empty set • universal set • Venn diagram

1. Read set notation.
2. Identify a region in a Venn diagram from set notation.
3. Use a Venn diagram to find cardinalities of sets.

6-C Probability of a Single Event — Thursday • 2/16

mutually exclusive • given • conditional probability

1. Use the size of the sample space to find the probability of an event.
2. Find the probability that either of two specific events will occur.
3. Find probabilities based on given information.

6-D Probability of Specific Multiple Events — Tuesday • 2/21

dependent events • independent events

1. Identify whether events are independent or dependent.
2. Find the probability of multiple events all occurring.

6-E Probability of General Multiple Events — Thursday • 2/23

binomail experiment

1. Calculate the probability of an event that can happen in different ways.
2. Use a complement to calculate a probability.
3. Calculate the probability of an event that can occur in different orders.
4. Explain the components of a binomial experiment calculation.
5. Calculate the probability of getting at most or at least *r* successes in a binomial experiment.

6-F Probability Distributions — Thursday • 3/2

probability distribution • expected value

1. Give the probability distribution of a simple event.
2. Make a histogram showing the probability distribution for a binomial experiment.
3. Calculate the expected value of a probability distribution.

6-A Counting Methods

A COMBINATION is a group of items not assigned specific labels or placements within the group. *N* CHOOSE *R*, written $\binom{n}{r}$ or nCr, is the number of combinations of r elements that can be made from a set of n elements. $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. Note that $\binom{n}{0} = 1$, $\binom{n}{1} = n$, and $\binom{n}{r} = \binom{n}{n-r}$.

❶ Count combinations.

1. Write out $n!$ in the numerator and write out $r!$ and $(n-r)!$ in the denominator.
2. Cancel the terms of $r!$ or the terms of $(n-r)!$ with the same terms in the numerator. (Or do not write these terms in the first place.)
3. Multiply and divide the remaining terms.

❶ In how many ways can Savannah choose her 3 favorite songs from a playlist of 9 songs?

$\binom{9}{3} = \frac{9!}{3!\cdot 6!} = \frac{9\cdot 8\cdot 7\cdot \cancel{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}{1\cdot 2\cdot 3\cdot \cancel{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}} = \frac{504}{6} = 84$

A SAMPLE SPACE consists of all the possible outcomes of an event, such as "♠, ♥, ♦, ♣" for suit of a card. The total number of possible outcomes of a series of events is the product of the sizes of the individual sample spaces.

The FUNDAMENTAL COUNTING PRINCIPLE states that if event A has a possible outcomes and event B has b possible outcomes, there are a total of ab possible outcomes in the sample space.

❷ Find the total number of possible outcomes in a series of events.

1. Identify the size of the sample space of each individual event, using $\binom{n}{r}$ as needed.
2. Multiple these sizes together.

❷ State the number of possible outcomes of the following.

a) Choose 3 representatives out of 9 seniors and 2 representatives out of 8 juniors.

$\binom{9}{3}\binom{8}{2} = 84 \cdot 28 = 2352$

b) Identify the 1st place, 2nd place, 3rd place, and 4th place finisher out of 25 racers.

$\binom{25}{1}\binom{24}{1}\binom{23}{1}\binom{22}{1} = 25 \cdot 24 \cdot 23 \cdot 22 = 303600$

The outcome of a series of events that each involve choosing one element from those remaining, as in example ❷b above, is called a PERMUTATION. This is the same as a combination, except that each item in the group is assigned a specific label or placement within the group. In other words, a combination is a selection of r items from a group of n items, and a permutation is a selection of r items, *chosen one at a time*, from a group of n items.

Permutations are never required, but can be used as an easier way to indicate mutliple combinations in which one of the remaining elements is chosen each time. For example, $\binom{9}{1}\binom{8}{1}\binom{7}{1}$ can be written as ${}_9P_3$.

The number of permutations of r elements that can be made from a set of n elements is $nPr = \frac{n!}{(n-r)!}$.

❸ Count permutations.

1. Write out $n!$ in the numerator and write out $(n-r)!$ in the denominator.
2. Cancel the terms of $(n-r)!$ with the same terms in the numerator. (Or do not write these terms in the first place.)
3. Multiply the remaining terms.

❸ In how many ways can Savannah choose her favorite, second favorite, and third favorite song from a playlist of 9 songs?

${}_9P_3 = \frac{9!}{6!} = \frac{9\cdot8\cdot7\cdot\cancel{6\cdot5\cdot4\cdot3\cdot2\cdot1}}{\cancel{6\cdot5\cdot4\cdot3\cdot2\cdot1}} = 504$

❹ Use a graphing calculator to count combinations and permutations.

1. Type the n value.
2. Push [MATH] and choose `PRB`.
3. Choose `nCr` or `nPr`.
4. Type the r value.

❹ Calculate $\binom{22}{19}$.

`22 nCr 19` = 1540

❺ Find the size of a sample space by using permutations, if possible.

1. If the elements in the group are indistinguishable, such as "three favorite songs," it is a combination.
2. If it matters which element is which, such as "favorite, second favorite, and third favorite song," it is a permutation.

❺ Use combinations to express the size of the sample space for each of the following. Then rewrite the solution using permutations if possible, or explain why not.

a) Claire chooses her 3 favorite months.

$\binom{12}{3}$ cannot be written as a permutation, because she is choosing a group of three equal items, not three seperate, distinguishable items.

b) Claire chooses her favorite, second favorite, and third favorite month.

$\binom{12}{1}\binom{11}{1}\binom{10}{1}$ can be written ${}_{12}P_3$.

c) Use a 6-sided die to choose a color for each of three teams.

$\binom{6}{1}\binom{6}{1}\binom{6}{1}$ cannot be written as a permutation, because the colors are not being removed as they are chosen.

d) Put 6 colors in a hat and draw three of them to choose a different color for each of three teams.

$\binom{6}{1}\binom{5}{1}\binom{4}{1}$ can be written ${}_6P_3$.

6-B Set Notation and Venn Diagrams

A SET is a collection of items, called ELEMENTS. $x \in A$ means x is an element of set A.

A SUBSET of another set is a set entirely contained within the other set. $A \subseteq B$ (or, equivalently, $A \supseteq B$) states that A is a subset of B (or, equivalently, that B contains A), meaning every element in A is also in B.

The CARDINALITY of a Set A, $|A|$, is the number of elements it contains. Other notations, such as $n(A)$, are sometimes used as well.

The INTERSECTION of Two Sets A and B, $A \cap B$, is the set of elements that are in both A and B.

The UNION of Two Sets A and B, $A \cup B$, is the set of elements that are in either A or B (or both).

The COMPLEMENT of a Set A , A' or A, is the set of elements not in A.

The UNIVERSAL Set, U, is the set of all elements within the given context.

The EMPTY Set, Ø, contains no elements.

❶ Read set notation.

1. Use the definitions above. In general, "∩" means *and*, "∪" means *or*, ' means *not*, and | | means *number of*.

❷ State the following in words, given A is the set of aces and B is the set of black cards.

a) A	the aces
b) U	all of the cards
c) Ø	none of the cards
d) $\|A\|$	the number of aces
e) $\|U\|$	the number of cards
f) $A \cup B$	the cards that are an ace or black
g) $A \cap B$	the cards that are an ace and black
h) $\|A \cap B\|$	the number of black aces
i) A'	the cards that are not aces
j) $(A \cap B)'$	the cards that are not black aces
k) $(A \cup B)'$	the cards that are not an ace or black

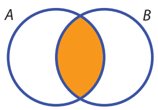
A VENN DIAGRAM shows up to three overlapping circles, with each circle representing a set.

❷ Identify a region in a Venn diagram from set notation.

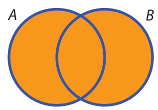
1. For an intersection, such as $A \cap B$, include only the space covered by both sets.
2. For a union, such as $A \cup B$, include all of each set.
3. For a complement, such as A', include everything but the set.

❷ Shade the following regions in a Venn diagram.

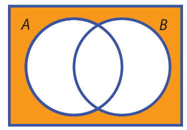
a) $A \cap B$
A and B



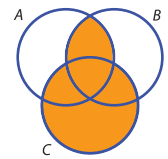
b) $A \cup B$
A or B



c) $(A \cup B)'$
not A or B



d) $C \cup (A \cap B)$
C, or A and B

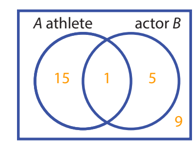


❸ Use a Venn diagram to find cardinalities of sets.

1. Label each known value.
2. Use subtraction to find the other values. In particular, $|A \cap B| = |A| + |B| - |A \cup B|$.

❸ In a class of 30 students, 16 are athletes, 6 are actors, and 9 are neither.

1. It is given that $|(A \cup B)'| = 9$ students are neither athletes nor actors.
2. $|A \cup B| = 30 - 9 = 21$ students are athletes or actors. (9 of the 30 are neither, so the rest are one or the other, or both.)
   $|A \cap B| = 16 + 6 - 21 = 1$ student is an athlete and an actor. (There are 16 athletes and 6 actors, for a total of 22. Since there are only 21 students who are athletes or actors, one of them must be both.)
   $|A \cap B'| = 16 - 1 = 15$ students are athletes but not actors. (There are 16 athletes. One is also an actor, and the rest are not.)
   $|A' \cap B| = 6 - 1 = 5$ students are actors but not athletes. (There are 6 actors. One is also an athlete, and the rest are not.)



6-C Probability of a Single Event

In probability, a set is used to represent an event, with the elements in the set being the event's possible outcomes.

The probability of event A, $P(A)$, is the probability of an outcome in A occuring. $P(A) = \frac{|A|}{|U|}$.

Because $|A|$ and $|U|$ have specific meanings, reducing a probability or converting it to a decimal or a percent causes information to be lost. Therefore, unless directed otherwise, use only fractions for probabilities in this class, and do not reduce probabilities other than those equal to 0 or 1.

❶ Use the size of the sample space to find the probability of an event

1. Identify the denominator $|U|$, the size of the sample space.
2. Identify the numerator $|A|$, the number of possible outcomes for event A within the sample space. Use methods from 6-B if needed.

❶ Eve draws two cards. Find the probability that…

a) the first card is a ace

1. There are $|U| = 52$ possible cards.
2. There are $|A| = 4$ possible aces.
3. $P(A) = \frac{4}{52}$

b) both cards are aces

There are $|U| = \binom{52}{2} = 1326$ possible combinations of 2 of the 52 cards.

There are $|A| = \binom{4}{2} = 6$ possible combinations of 2 of the 4 aces.

$P(A) = \frac{6}{1326}$

Two events are said to be MUTUALLY EXCLUSIVE, or Disjoint, if the occurance of one eliminates the possibility of the other, such as clubs and hearts on a single card. If A and B are mutually exclusive, the probability that one of them will happen is $P(A \cup B) = P(A) + P(B)$.

The above formula does not work for events that are not mutually exclusive, because some outcomes would be counted more than once. To account for this, the overlap between the events can be subtracted: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

❷ Find the probability that either of two specific events will occur.

1. Add the probabilities of the two events.
2. Subtract the probability of the two events happening simultaneously, since this has been double-counted.

❷ Find the probability of a card being as stated.

a) red or an ace

1. $\frac{26}{52} + \frac{4}{52} = \frac{30}{52}$
2. $\frac{30}{52} - \frac{2}{52} = \frac{28}{52}$

b) a 9 or an ace

$\frac{4}{52} + \frac{4}{52} = \frac{8}{52}$

$\frac{8}{52} - 0 = \frac{8}{52}$

Probability is based on what is known, not on what has happened. GIVEN means *known* (or *knowing*). Given information changes the universal set. For example, given a card is red, the probability of it being hearts is $\frac{13}{26}$, because U now only includes the 26 red cards.

CONDITIONAL Probabilities incorporate given information that would not necessarily be assumed, such as *the card is red*. The probability of event A after knowledge of event B has been taken into account is the probability of A given B: $P(A \mid B) = \frac{|A \cap B|}{|B|}$.

❸ Find probabilities based on given information.

1. Do not assume any information, even for events that have already occurred.
2. Take into account all known (given) information, even for events that have not yet occurred.

❸ Find the following probabilities for Eve's two cards.

a) The second card is an ace.

1. The first card remains unknown. Do no assume it is or is not an ace. There are still $|U| = 52$ possible outcomes, $|A| = 4$ of which are aces, so $P(A) = \frac{4}{52}$.

b) The second card is an ace, given the first card is an ace.

2. The first card is known, so use that information. There are only $|U| = 51$ remaining possible outcomes for the second card, $|A| = 3$ of which are aces, so $P(A) = \frac{3}{51}$.

c) The first card is an ace, given the second card will be an ace.

2. The second card is known even though it hasn't been flipped yet, so use that information. There are only $|U| = 51$ remaining possible outcomes for the first card, $|A| = 3$ of which are aces, so $P(A) = \frac{3}{51}$.

6-D Probability of Specific Multiple Events

The probabilities of DEPENDENT Events are conditional: They are influenced by each other's outcome.

The probabilities of INDEPENDENT Events do not change.

❶ Identify whether events are dependent or independent.

1. If the probability of an event is the same every time, it is independent.
2. If the outcome of an event changes the sample space for other events, it is dependent.

❶ Ashley rolls three dice, trying to roll 6's, and she draws two cards, trying to draw kings.

1. The die rolls are independent: Each roll has a $\frac{1}{6}$ chance of being a 6, regardless of the other rolls.
2. The card draws are dependent: After she knows the first card, there are only 51 remaining possibilities for the second card. The second card has a $\frac{4}{51}$ chance of being an ace if the first card was not an ace, or a $\frac{3}{51}$ chance of being an ace if the first card was an ace.

The probability of both A and B occuring is the product of their probabilities: $P(A \cap B) = P(A) \bullet P(B)$. If the events are dependent, the probability of one of them must be adjusted to its conditional probability based on the other event: $P(A \cap B) = P(A) \bullet P(B \mid A)$, or equivalently, $P(B) \bullet P(A \mid B)$.

❷ Find the probability of multiple events all occurring.

1. List what has to occur. Ignore events that do not matter.
2. Identify the conditional probability of each event, given each of the previous events occuring.
3. Multiply all the probabilities together.

❷ James draws five cards. Calculate the probability that the first two cards are aces and the fourth card is not an ace.

1. first card ace, second card ace, fourth card not ace
   (Note that the third and fifth card are irrelevant to this problem.)
2. $P(\text{first card is an ace}) = \frac{4}{52}$
   $P(\text{second card is an ace, given first card is an ace}) = \frac{3}{51}$
   $P(\text{fourth card is not an ace, given first two cards are aces}) = \frac{48}{50}$
3. $\frac{4}{52} \bullet \frac{3}{51} \bullet \frac{48}{50} = \frac{576}{132600}$

❷ Janaya rolls three 6-sided dice. Calculate the probability that they all roll the same number.

1. second roll same as first, third roll same as first
   (Note that it doesn't matter what the first roll is.)
2. $P(\text{second roll is same as first}) = \frac{1}{6}$
   $P(\text{third roll is same as first}) = \frac{1}{6}$
3. $\frac{1}{6} \bullet \frac{1}{6} = \frac{1}{36}$

6-E Probability of General Multiple Events

Some events can take place in different ways. To calculate the total probability, the probability of each of the possible ways is added together.

❶ Calculate the probability of an event that can occur in different ways.

1. Identify each different way it can occur.
2. Calculate the probability of each way (see 6-D).
3. Add these probabilities together.

❶ Hanna grabs 3 random pens from a drawer with 6 black pens, 4 red pens, and 1 purple pen. What is the probability that they are all the same color?

1. It could be black, black, black, or it could be red, red, red.
2. $P(\text{all black}) = \frac{6}{11} \bullet \frac{5}{10} \bullet \frac{4}{9} = \frac{120}{990}$
   $P(\text{all red}) = \frac{4}{11} \bullet \frac{3}{10} \bullet \frac{2}{9} = \frac{24}{990}$
3. $P(\text{all black or all red}) = \frac{120}{990} + \frac{24}{990} = \frac{144}{990}$

An event must either happen or not happen. Therefore, these two possibilities are complements of each other, making the sum of their probabilities 100%: $P(A) + P(A') = 1$.

Often it is simpler to calculate $P(A')$ than $P(A)$, especially if when A can happen in more ways than A' can. In this case, $P(A)$ can be found by subtracting its complement from 1: $P(A) = 1 - P(A')$.

❷ Use a complement to calculate a probability.

1. Identify the complement A' of the event A.
2. Calculate the probability of the complement $P(A)$.
3. Subtract $P(A)$ from 1.

❷ What is the probability that at least one of the three pens Hanna grabs, above, is red?

1. If she doesn't grab at least one red pen, then none of the pens she grabs are red.
2. $P(R', R', R') = \frac{7}{11} \bullet \frac{6}{10} \bullet \frac{5}{9} = \frac{210}{990}$
3. $P(\text{at least one red}) = \frac{990}{990} - \frac{210}{990} = \frac{780}{990}$

When the different ways in which an event can happen are simply different orders of the same thing, each way will have the same probability. Therefore, instead of separately identifying each order and calculating its probability, the probability can be calculated once and multiplied by the number of possible orders.

❸ Calculate the probability of an event that can occur in different orders.

1. Identify one possible order.
2. Calculate the probability of this order.
3. Count or calculate the number of possible orders, using $\binom{n}{r}$ as needed.
4. Multiply the number of possible orders by the probability of each order.

❸ David rolls five 6-sided dice. What is the probability that exactly two of them roll '6'?

1. One possible order is that the first two roll '6' and the other three do not.
2. $P(6, 6, 6', 6', 6') = \frac{1}{6} \bullet \frac{1}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} = \frac{125}{7776}$
3. There are five dice to choose from for which two roll '6', so there are $\binom{5}{2} = 10$ possible orders.
4. $P(\text{all different colors}) = 10 \bullet \frac{125}{7776} = \frac{1250}{7776}$

For a series of n trials which have the same probability of success p each time and same probability of failure $q = 1 - p$ each time, the probability of exactly r successes is $P(r) = \binom{n}{r}p^r q^{n-r}$. This is called a BINOMIAL EXPERIMENT.

❹ Explain the components of a binomial experiment calculation.

1. p^r is the probability of r successful trials, each with probability p of success.
2. q^{n-r} is the probability of $n - r$ unsuccessful trials, each with probability q of failure.
3. $\binom{n}{r}$ is the number of orders for r of the n trials to be successes and the rest to be failures. Each order has probability $p^r q^{n-r}$.
4. $\binom{n}{r}p^r q^{n-r}$ is the probability that one of the $\binom{n}{r}$ orders will occur (that is, that exactly r of the n trials will be successes.

❺ Calculate the probability of getting at most or at least r successes in a binomial experiment.

1. Identify n, p, and q (see above).
2. Identify the values of r that satisfy the given scenario.
3. Do step 5, above, for each value of r in the range stated or for the complement (each value of r not in the range stated), whichever is easier.
4. Add together the results of step 3.
5. If you found the complement instead of the actual probability, subtract it from 1.

❺ Find the probability that out of five 6-sided dice, fewer than four will roll '6'.

1. $n = 5$, $p = \frac{1}{6}$, $q = \frac{5}{6}$
2. Fewer than 4 out of 5 means $r = 0, 1, 2$ or 3.
3. For the complement, $r = 4$ or 5, which is less work to calculate than $r = 0, 1, 2,$ or 3.

   $P(4) = \binom{5}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^1 = 5\left(\frac{1}{1296}\right)\left(\frac{5}{6}\right) = \frac{25}{7776}$

   $P(5) = \binom{5}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^0 = 1\left(\frac{1}{7776}\right)(1) = \frac{1}{7776}$
4. $P(4 \text{ or } 5) = \frac{25}{7776} + \frac{1}{7776} = \frac{26}{7776}$
5. $P(< 4) = \frac{7776}{7776} - \frac{26}{7776} = \frac{7750}{7776}$

6-F Probability Distributions

A PROBABILITY DISTRIBUTION shows all the possible outcomes of an event and how likely each one is. The sum of the probabilities in a probability distribution is 1 (that is, 100%) because it includes all possibilities.

❶ Give the probability distribution of a simple event.

1. List each possible outcome.
2. State the probability of each of these outcomes.

❶ Show the probability distribution for a coin flip.

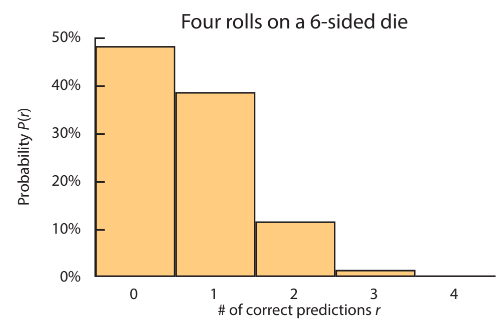
1. heads tails
2. 50% 50%

❷ Make a histogram showing the probability distribution for a binomial experiment.

1. Identify the number of trials n, the probability of success p, and the probability of failure $q = 1 - p$.
2. Label the y-axis $P(r)$.
3. Label the x-axis as number of successes, and scale it from 0 to n. Be specific to the distribution; for example, write *number of correct predictions* or *number of 6's* rather than *number of successes* without saying what a success is.
4. For each value of r from 0 to n, find the value of $P(r) = \binom{n}{r} p^r q^{n-r}$ as a percentage.
5. Make a bar for each value of r to show the probability.
6. Title the graph in a way that, combined with the x-axis label, clearly conveys the experiment, such as *rolling eight 6-sided dice.*

❷ Predict four rolls of a 6-sided die.

1. $n = 4$ trials, $p = \frac{1}{6}$ chance of a correct prediction, $q = \frac{5}{6}$ chance of an incorrect prediction
4. $P(0) = \binom{4}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^4 = 48.2\%$
   $P(1) = \binom{4}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^3 = 38.6\%$
   $P(2) = \binom{4}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 = 11.6\%$
   $P(3) = \binom{4}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^1 = 1.5\%$
   $P(4) = \binom{4}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^0 = 0.1\%$



The EXPECTED VALUE of a probability distribution is the theoretical average result.

❸ Calculate the expected value of a probability distribution.

1. Identify the probability of each outcome. Events with a value of 0 can be ignored.
2. Multiply the probability of each outcome by its value.
3. Add these products together.

❸ A 10-space spinner has 6 blue spaces worth 10 points each, 3 red spaces worth 25 points each, and one black space worth -100 points.

Event	Probability	Value	Product
blue	$\frac{6}{10}$	10	6.0
red	$\frac{3}{10}$	25	7.5
black	$\frac{1}{10}$	-100	-10.0
			3.5

Spins are worth 3.5 points each, on average.

For binomial distributions, the expected value can be calculated by $\mu = np$.