Chapter Five

Exponential and Logarithmic Functions

Exponential Functions

Logarithmic Functions

Properties of Logarithms

Exponential Equations

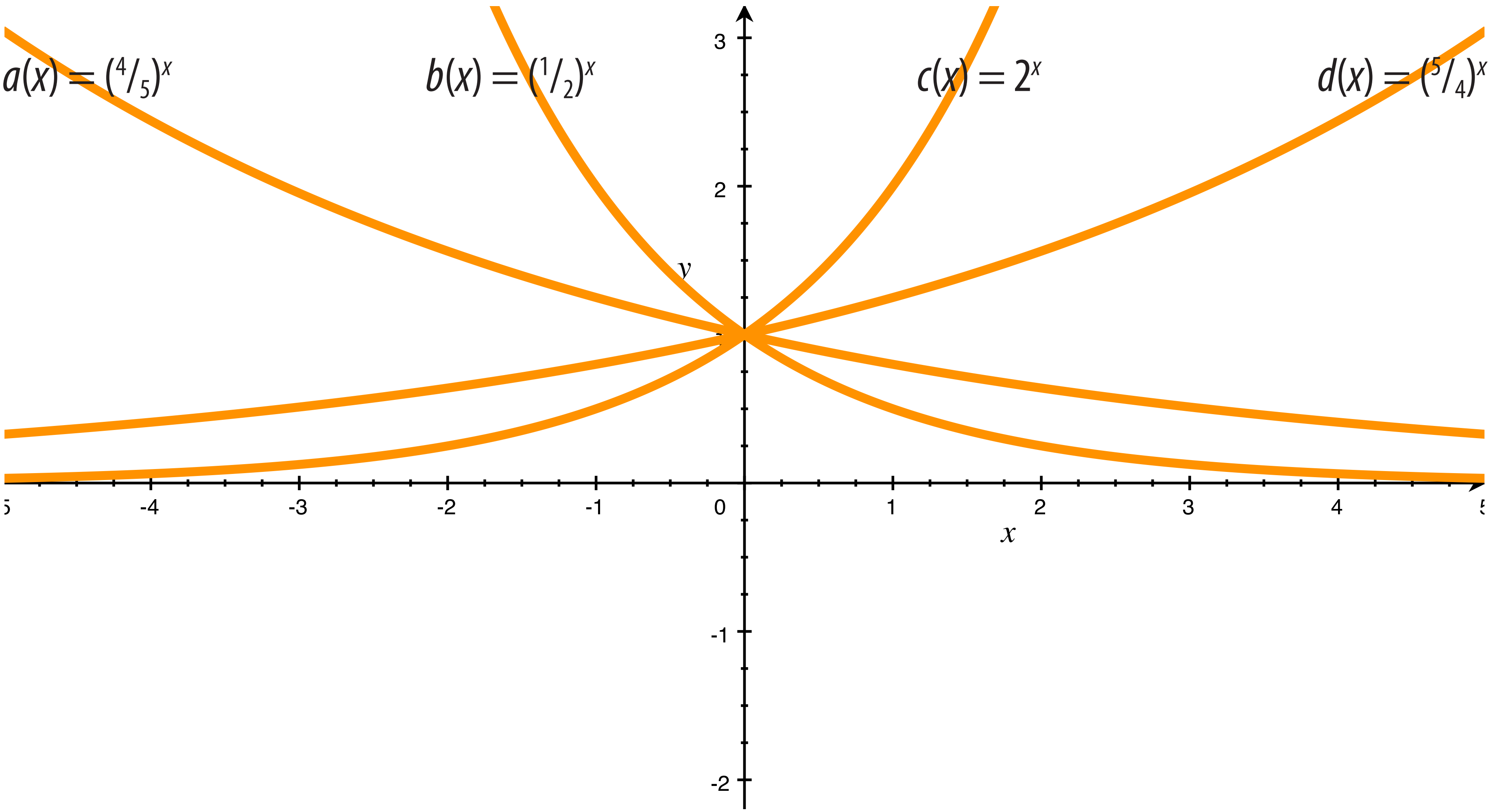
Exponential Situations

Logarithmic Equations

Exponential Functions

In an **exponential function**, the variable is in the exponent: $f(x) = b^x$.

The base b can be any positive number other than 1. The closer b is to 1, the closer the graph is to a horizontal line. The graph increases if b is greater than 1, or decreases if b is less than 1.



Exponential Growth and Decay

A **factor** is a value being multiplied. In $f(x) = b^x$, the base b is a factor.

If b is above 1, it is a **growth factor**, and the amout it is above 1 is the **growth rate**.

If b is below 1, it is a **decay factor**, and the amout it is below 1 is the **decay rate**.

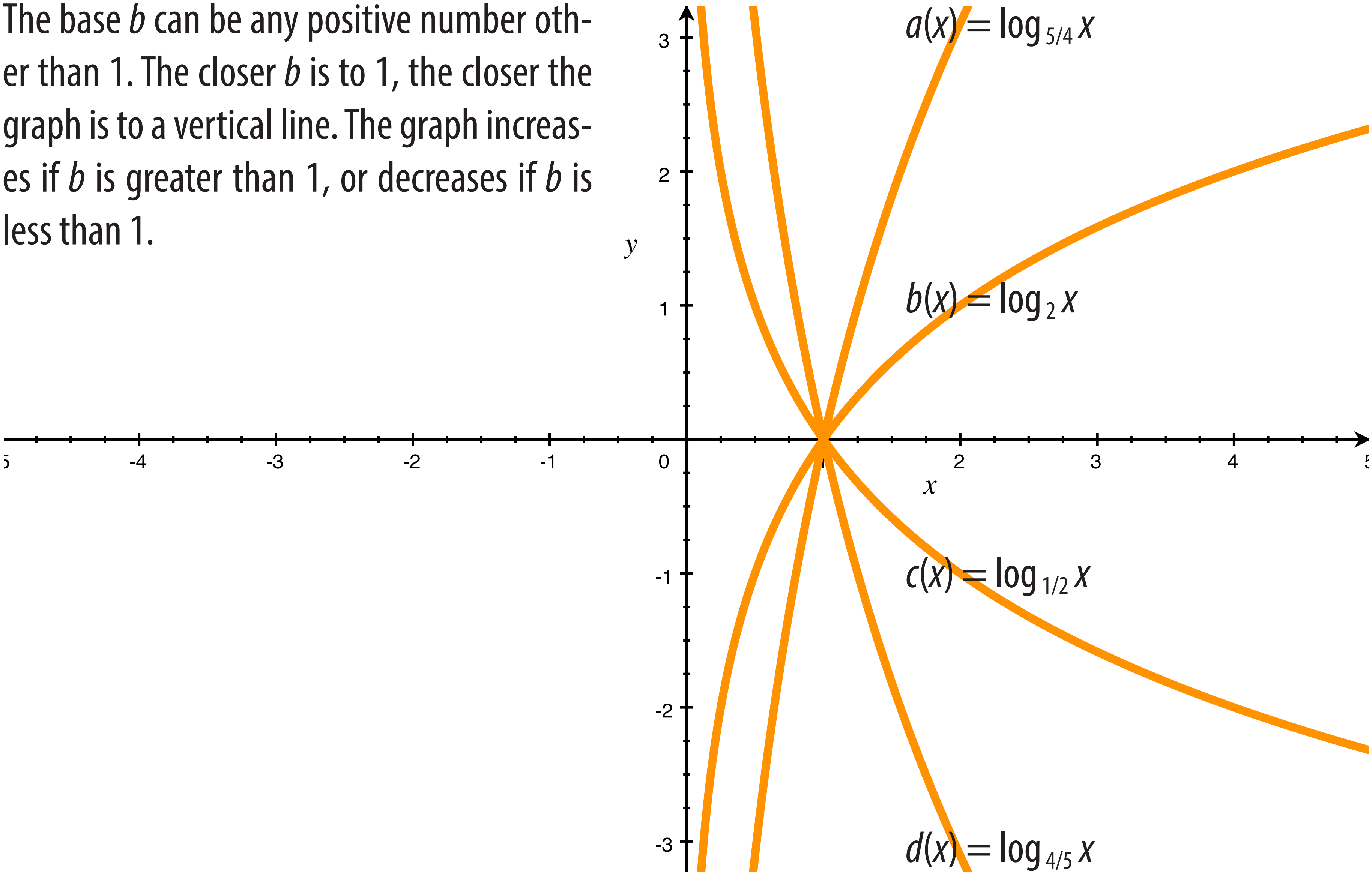
Change, in words	Growth Rate (r) or Decay Rate (r)	Growth Factor ($b = 1 + r$) or Decay Factor ($b = 1 - r$)
8% increase	.08	1.08
8% decrease	.08	0.92
.081% decrease	.00081	0.99919
increase by half	.5	1.5
increase by half a percent	.005	1.005
75% more	.75	1.75
175% more	1.75	2.75
tripling	2	3

A common mistake is to forget to add 1 to growth rates when they are large such as 175%. $f(x) = 1.75^x$ represents only 75% growth. It should be $f(x) = 2.75^x$ to represent 175% growth.

Logarithmic Functions

The inverse of an exponential function is a **logarithmic function**: $f(x) = \log_b x$.

The base b can be any positive number other than 1. The closer b is to 1, the closer the graph is to a vertical line. The graph increases if b is greater than 1, or decreases if b is less than 1.



Logarithms

A logarithm is an exponent. Specifically, the **logarithm** with base *b* of a number *x* is the exponent needed to change *b* into *x*. A simple logarithmic equation can be rewritten as an exponential equation by expressing the value of the logarithm as an exponent: $\log_b m = n \Leftrightarrow b^n = m$. Some examples are shown at below.

Logarithmic form	Exponential form	Value
$\log_4 16 = x$	$4^x = 16$	$x = 2$
$\log_{16} 4 = x$	$16^x = 4$	$x = ½$
$\log_4 {}^1/_{16} = x$	$4^x = {}^1/_{16}$	$x = -2$
$\log_4 -16 = x$	$4^x = -16$	not possible

The most common logarithm bases are 2, *e*, and 10. The irrational number $e \approx 2.718$ is used extensive-ly in calculus.

Name	Base	Notation
binary log	2	$\log_2 x$
natural log	e	$\ln x$
common log	10	$\log x$

Properties of Logarithms

Property	Rule	Example
Product	$\log_b x + \log_b y = \log_b xy$	$\log_6 4 + \log_6 9 = \log_6 36 = 2$
Quotient	$\log_b x - \log_b y = \log_b \frac{x}{y}$	$\log_2 88 - \log_2 11 = \log_2 8 = 3$
Power	$\log_b x^y = y \log_b x$	$\log_2 8^5 = 5 \log_2 8 = 5 \cdot 3 = 15$
Negative	$\log_b \frac{1}{x} = -\log_b x$	$\log_2 \frac{1}{16} = -\log_2 16 = -4$
Reciprocal	$\log_b x = \frac{1}{\log_x b}$	$\log_{16} 2 = \frac{1}{\log_2 16}$
Change of Base	$\log_b x = \frac{\log_a x}{\log_a b}$	$\log_8 16 = \frac{\log_2 16}{\log_2 8} = \frac{4}{3}$

Solving simple exponential and logarithmic equations

Like all equations, exponential and logarthmic equations can be solved by applying their inverse to each side. **Exponentiation** is making an expression an exponent of a base.

Type of equation	Approach	Example	Inverse applied	Solution
Exponential	Take the log of each side, using the base used for the exponential expression.	$2^x = 64$	$\log_2 2^x = \log_2 64$	$x = 6$
Logarithmic	Exponentiate each side, using the base used for the logarithmic expression.	$\log_2 x = 6$	$2^{\log_2 x} = 2^6$	$x = 64$

Solving complicated exponential equations

An exponential expression in an equation should be isolated before its inverse is applied. Then the equation can be solved by taking the log of each side, simplifying, and algebraic manipulation.

$10(2^{5x-6}) + 200 = 5000$	Step	Comment
$2^{5x-6} = 480$	Subtract 200 from each side, and divide each side by 10.	This isolates the exponential expression.
$\log_2 2^{5x-6} = \log_2 480$	Take the log of each side.	Use the same base (2) as the exponential.
$5x - 6 = \log_2 480$	Simplify.	Log with base 2 cancels an exponential with base 2.
$5x - 6 = \frac{\log 480}{\log 2}$	Rewrite $\log_2 480$ using the change of base property.	Change the base to 10.
$5x - 6 \approx 8.91$	Evaluate the log.	Base 10 logs can be evaluated directly with a calculator.
$x \approx 2.98$	Add 6 to each side, and divide each side by 5.	Finish solving with basic algebra.

Solving equations with two exponential expressions

If both sides of an equation are exponential, the common or natural log can be taken on each side, allowing for simplification using the power property.

$2^{5x+9} = 20^{2x}$	Step
$\log 2^{5x+9} = \log 20^{2x}$	Take the log on each side.
$(5x + 9) \log 2 = 2x \log 20$	Apply the power property on each side.
$(5x + 9)(.30) \approx 2x(1.30)$	Use a calculator to evaluate the logs.
$1.5x + 2.7 \approx 2.6x$	Distribute.
$x \approx 2.45$	Finish solving with basic algebra.

Common mistakes with exponentials and logarithms

Scenario	Common Mistake	Issue
Solve $3^x = 9$	$\log_3 3^x = \log_3{}^9$	Bases must be clearly subscript. Arguments must not be superscript. $\text{Log}_3 9$ is 2, but the way it is written, $\log_3{}^9$ is log 19683, which is 4.3.
Simplify $\log x - \log 3y$	$\frac{\log x}{\log 3y}$	The quotient property is different from the change of base property. $\log x - \log 3y$ is $\log \frac{x}{3y}$. $\text{Log}_{3y} x$ is $\frac{\log x}{\log 3y}$.
Solve $8(½)^x = 6$	$\log_4 4^x = \log_4 6$	Only the ½, not the 8, is to the power of x, so 8 cannot be multiplied in. Each side must be divided by 8 before the log is taken.
Solve $x^{200} = 18$	$x = 18^{1/200} = 1.01$	A growth factor of 1.01 represents a growth rate of .01, which has only one sigificant digit. This creates huge rounding error. $1.01^{200} = 7.3$, not 18.
Simplify $3 \log 4x$	$\log 4x^3$	$4x$ to the 3rd power is $(4x)^3$, not $4x^3$. For the power property, the power is the exponent of the whole argument, not just the variable.
Simplify $\log 64x^3$	$3 \log 64x$	This is the same error as above, in reverse. The expression should be $3 \log 4x$.

Exponential situations

The equation $f(x) = ab^x$ can be used to model exponential situations. Keep in mind that $b = 1 + r$ for growth and $b = 1 - r$ for decay.

In the example below, 500 grams (a) of a substance that decays at a rate of 8% per day (r) will decay down to 41 grams ($f(x)$) in 30 days (x).

Variable	Meaning	How to solve for	Example
a	starting amount (amount at time 0)	Divide each side by b^x.	$41 = a(.92)^{30}$ $500 = a$
b	growth or decay factor	Divide each side by a, and then take the power of $\frac{1}{x}$ on each side.	$41 = 500(b)^{30}$ $.082 = b^{30}$ $.082^{1/30} = b$ $.92 = b$
x	time	Divide each side by a, and then take the log on each side using base b.	$41 = 500(.92)^x$ $.082 = .92^x$ $\log_{.92} .082 = \log_{.92} .92^x$ $30 = x$
$f(x)$	ending amount (amount at time x)	Evaluate ab^x.	$f(30) = 500(.92)^{30}$ $f(30) = 41$

Solving complicated logarithmic equations

Like exponential expressions, a logarithmic expression in an equation should be isolated before its inverse is applied.

If there is more than one logarithmic expression, properties of logarithms can be used to combine them into a single logarithmic expression.

The change of base property can be used to make the bases the same if needed.

$18\log_8 3x - \log_2 6x = 16$	Step
$18\frac{\log_2 3x}{\log_2 8} - \log_2 6x = 16$	Apply the change of base property.
$18\frac{\log_2 3x}{3} - \log_2 6x = 16$	Evaluate $\log_2 8$.
$6\log_2 3x - \log_2 6x = 16$	Simplify.
$\log_2 (3x)^6 - \log_2 6x = 16$	Apply the power property.
$\log_2 \frac{(3x)^6}{6x} = 16$	Apply the quotient property.
$\log_2 121.5x^5 = 16$	Simplify the quotient.
$2^{\log_2 121.5x^5} = 2^{16}$	Exponentiate each side with base 2.
$121.5x^5 = 65{,}536$	Simplify and evaluate.
$x \approx 3.52$	Divide each side by 121.5, and then take the 5^{th} root of each side.