CHAPTER FIVE: EXPONENTIAL AND LOGARITHMIC FUNCTIONS — Review January 30 ⟿ Test February 7

An exponential function is one with the independent variable in the exponent, such as $f(x) = 3^{5x}$. A logarithm is an exponent on a specified base. For example, the log with base 3 of 3^{5x}, written "$\log_3 3^{5x}$" is 5x. Logarithmic functions are the inverse of exponential functions and can be used to solve exponential equations. For example, if $3^{5x} = 90$, applying $\log_3$ on each side undoes the exponential function on the left leaving just the exponent, 5x. The other side, $\log_3 90$, is a real number that can be divided by 5 to solve for x.

5-A Exponential Functions — Monday • 1/9

exponential function • exponential growth and decay • compound interest

❶ Identify a scenario's rate of increase or decrease and its growth or decay factor.
❷ Sketch $f(x) = b^x$.
❸ Write a function for an exponential growth or decay situation, and use it to calculate future values.

5-B Logarithmic Functions — Tuesday • 1/10

logarithm • common log • natural log

❶ Simplify the composition of a logarithmic and an exponential function.
❷ Evaluate a simple logarithm by hand.
❸ Calculate a common or natural logarithm with a calculator.
❹ Sketch $f(x) = \log_b x$.

5-C Properties of Logarithms — Thursday • 1/12

product property • quotient property • power property • negative property • reciprocal property• change of base property

❶ Simplify a logarithmic expression.
❷ Evaluate a logarithm in which the base and the argument are powers of the same number.
❸ Use a calculator to evaluate any logarithm.

5-D Exponential Equations — Thursday • 1/19

❶ Solve an exponential equation by using log with a base in the equation.
❷ Solve an exponential equation by using common log or natural log.
❸ Solve an exponential equation without a calculator.

5-E Exponential Situations — Tuesday • 1/24

half-life

❶ Translate a description of an exponential situation into an equation, and solve it.

5-F Logarithmic Equations — Thursday • 1/26

exponentiate

❶ Solve a logarithmic equation.

5-A Exponential Functions

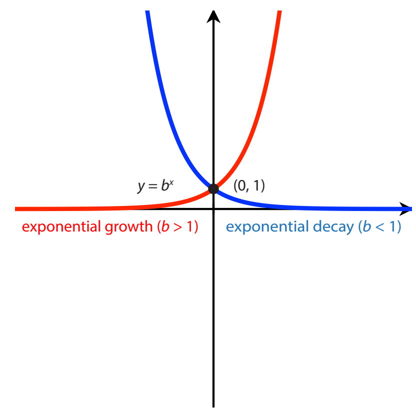
An EXPONENTIAL Function is one in which the independent variable is in the exponent: $f(x) = b^x$.

In Exponential GROWTH, $b = 1 + r$ is called the GROWTH FACTOR, and r is the rate of increase. If r is negative, b will be less than one and $-r$ is the rate of decrease; this is called Exponential DECAY.

An increase is an addition, a decrease is a subtraction, and a factor is a multiplier. Therefore, if a value is increasing by x or is x times more, then r is x, if a value is decreasing by x then r is $-x$, and if a value is x times as much then b is x.

❶ Identify a scenario's rate of increase or decrease and its growth or decay factor.

1. Change the given value to a decimal if needed.
2. If the change given is an increase of x, then $r = x$ and $b = 1 + r$.
   If the change given is a decrease of x, then $r = -x$ and $b = 1 + r$.
   If the change given is a factor of x (e.g., "x times as much") then $b = x$ and $r = b - 1$.



❶ a) 4% less
$r = -.04$
$b = 1 + -.04 = .96$

b) 200% more
$r = 2$
$b = 2 + 1 = 3$

c) 3 times as much
$b = 3$
$r = 3 - 1 = 2$

d) 3 times more
$r = 3$
$b = 3 + 1 = 4$

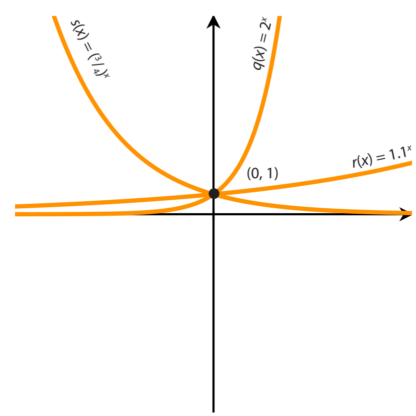
The graph of $f(x) = b^x$ is asymptotic to the x-axis. The domain is all real numbers, and the range is all positive real numbers. Since $b^0 = 1$ for all possible values of b, the graph always passes through the point $(0, 1)$.

❷ Sketch $f(x) = b^x$.

1. Plot the point $(0, 1)$.
2. The further b is from 1, the faster the curve changes direction away from the x-axis and toward the y-axis. If $b = 1$, the curve is perfectly horizontal (and thus not actually exponential).
3. For exponential growth $(b > 1)$, sketch the curve up to the right from $(0, 1)$ and asymptotic to the x-axis before this point, as shown by the red curve above.
   For exponential decay $(b < 1)$, sketch the curve up to the left from $(0, 1)$ and asymptotic to the x-axis after this point, as shown by the blue curve above.

❷ Compare the graphs of $q(x) = 2^x$, $r(x) = 1.1^x$, and $s(x) = \left(\frac{3}{4}\right)^x$.

1. All three graphs pass through the point $(0, 1)$.
2. q bends the most to follow the y-axis. r barely bends at all, instead being almost a horizontal line.
3. q and r are exponential growth, curving upward. s is exponential decay, curving downward.



$f(x) = ab^x$ represents a quantity $f(x)$ at time x, given a growth rate of $r = b - 1$ and an initial quanity of a at time $x = 0$.

❸ Write a function for an exponential growth or decay situation, and use it to calculate future values.

1. Identify the initial amount a.
2. Identify the growth or decay factor b.
3. Fill in a and b in the formula $f(x) = ab^x$.
4. Identify the units for the independent variable x.
5. Determine x, the amount of time past the starting point. x will be negative if for a time before the starting point.
6. Plug in x.

❸ Annual population growth in California has been 0.87% over the past six years, reaching 39.3 million residents in 2016. Use this growth rate to calculate an estimate of California's population in the following years.

a) 2025 b) 2006

1. $a = 39.3$
2. $b = 1 + .0087 = 1.0087$
3. $f(x) = 39.3(1.0087)^x$
4. x = # of years after 2016
5. $x = 2025 - 2016 = 9$ $x = 2006 - 2016 = -10$
6. $f(9) = 39.3(1.0087)^9 = 42.5$ million $f(-10) = 39.3(1.0087)^{-10} = 36.0$ million

5-B Logarithmic Functions

A logarithm is an exponent: The LOGARITHM with base b of a number x is the exponent that will change b into x.

$\log_b x = y$ means $x = b^y$

$f^{-1}(x) = \log_b x$ is the inverse of $f(x) = b^x$. Since inverses cancel each other (that is, $f^{-1}(f(x)) = f(f^{-1}(x))$), $\log_b b^x = x$, and $b^{\log_b x} = x$.

In real-world contexts, the most frequently used bases of logs are 2, 10, and e, where $e \approx 2.71828$ is an irrational number of great mathematical importance, especially in calculus.

$\log_{10} x$ is called the COMMON Log of x and is written "log x".

$\log_e x$ is called the NATURAL Log of x and is written "ln x".

❶ Simplify the composition of a logarithmic function and an exponential function.

1. If not already done, write the expression so that the exponential function and the logarithmic function have the same bases.
2. Cancel the exponential function with the logarithmic function.

❶ Simplify the following expressions.

a) $\log_6 6^{2x}$ — $2x$

b) $3^{\log_3 2x}$ — $2x$

c) $\ln e^9$ — 9

d) $e^{\ln 9}$ — 9

e) $\log_2 8^x$ — $\log_2 (2^3)^x = \log_2 2^{3x} = 3x$

❷ Evaluate a simple logarithm by hand.

1. Change the problem from logarithmic form ($\log_b x = y$) to exponential form ($x = b^y$) and determine the exponent that makes the statement true.

❷ Evaluate the following logarithms if possible.

a) $\log_4 16$ — $4^2 = 16$

b) $\log_4 \frac{1}{16}$ — $4^{-2} = \frac{1}{16}$

c) $\log_4 2$ — $4^{½} = 2$

d) $\log_4 \frac{1}{2}$ — $4^{-½} = \frac{1}{2}$

e) $\log_4 4$ — $4^1 = 4$

f) $\log_4 1$ — $4^0 = 1$

g) $\log_4 0$

h) $\log_4 -16$

not possible: 4^x cannot be 0 or negative

Common log and natural log have their own calculator buttons.

❸ Evaluate a common or natural logarithm with a calculator.

1. Use the [LOG] button for base 10.
   Use the [LN] button for base e.

❸ Evaluate the following logarithms if possible.

a) log 1000 — 3

b) log .01 — -2

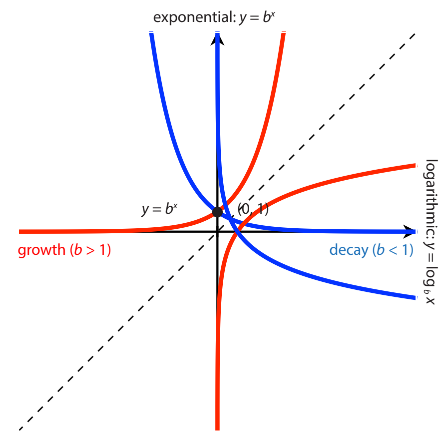
c) log 2.24 — 0.35

d) ln 20 — 3.22

e) log -100 — nonreal answer

The basic logarithmic function is $f(x) = \log_b x$. Its graph is asymptotic to the y-axis.

Since they are inverses, the domain of $f(x) = \log_b x$ is the same as the range of $f^{-1}(x) = b^x$, which is positive real numbers. Likewise, the range of $f(x) = \log_b x$ is the same as the domain of $f^{-1}(x) = b^x$, which is all real numbers. Since $\log_b 1 = 0$ for all possible values of b, the graph always passes through the point $(1, 0)$.

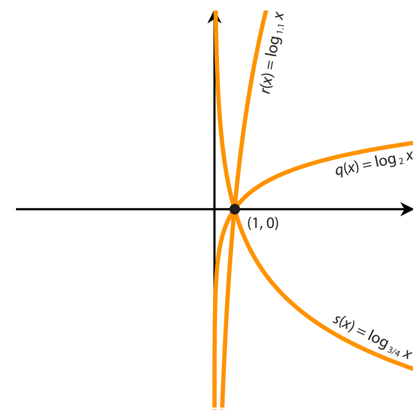


❹ Sketch $f(x) = \log_b x$.

1. Plot the point $(1, 0)$.
2. The further b is from 1, the faster the curve changes direction away from the y-axis and toward the x-axis. If b were exactly 1 (which is not possible), the curve would be perfectly vertical.
3. For logarithmic growth $(b > 1)$, sketch the curve up to the right from $(1, 0)$ and asymptotic to the y-axis below this point, as shown by the red curve on the right.
   For logarithmic decay $(b < 1)$, sketch the curve up to the left from $(1, 0)$ and asymptotic to the x-axis below this point, as shown by the blue curve on the right.

❹ Compare the graphs of $q(x) = \log_2 x$, $r(x) = \log_{1.1} x$, and $s(x) = \log_{3/4} x$.

1. All three graphs pass through the point $(1, 0)$.
2. q bends the most to follow the x-axis. r barely bends at all, instead being almost a vertical line.
3. q and r are logarithmic growth, curving upward. s is logarithmic decay, curving downward.



5-C Properties of Logarithms

The following Properties of Logarithms apply to all positive numbers except $b = 1$.

Property	Rule	Example	Related Property
Product	$\log_b x + \log_b y = \log_b xy$	$\log 25 + \log 4 = \log 100 = 2$	$b^x b^y = b^{x+y}$
Quotient	$\log_b x - \log_b y = \log_b \frac{x}{y}$	$\log 6000 - \log 6 = \log 1000 = 3$	$\frac{b^x}{b^y} = b^{x-y}$
Power	$\log_b x^y = y \log_b x$	$\log 1000^2 = 2 \log 1000 = 2 \bullet 3 = 6$	
Negative	$\log_b \frac{1}{x} = -\log_b x$	$\log_2 \frac{1}{8} = -\log_2 8 = -3$	$\frac{1}{b^x} = b^{-x}$
Reciprocal	$\log_b x = \frac{1}{\log_x b}$	$\log_8 2 = \frac{1}{\log_2 8} = \frac{1}{3}$	
Change of Base	$\log_b x = \frac{\log_a x}{\log_a b}$	$\log_{100} 1000 = \frac{\log 1000}{\log 100} = \frac{3}{2}$	

❶ Simplify a logarithmic expression.

1. Use the above properties.

❶ Simplify the following expressions. State the property of logarithms used for each step.

a) $2 \log_3 x + \log_3 10$

$\log_3 x^2 + \log_3 10$	power property
$\log_3 10x^2$	product property

b) $\log_3 9^{4x}$

$4x \bullet \log_3 9$	power property
$4x \bullet 2 = 8x$	definition

c) $\log_3 \left(\frac{1}{9}\right)^{4x}$

$4x \bullet \log_3 \frac{1}{9}$	power property
$4x \bullet -\log_3 9$	negative property
$4x \bullet -2 = -8x$	definition

The change of base property is commonly used to change to base 10 or base e, since those can be evaluated on a calculator. It also is used to change between two bases that are powers of the same number, such as $9 = 3^2$ and $27 = 3^3$.

❷ Evaluate a logarithm in which the base and the argument are powers of the same number.

1. Identify a common base.
2. Apply the change of base property, using the common base as the new base.

❷ $\log_{32} 8$

1. 8 and 32 are both powers of 2.
2. $\frac{\log_2 8}{\log_2 32} = \frac{3}{5}$

❸ Use a calculator to evaluate any logarithm.

1. Apply the change of base property using base 10 as the new base, by typing `log(x)/log(b)`.

❸ $\log_3 20$

1. `log(20)/log(3)` ≈ 2.73

5-D Exponential Equations

Since logarithms are the inverse of exponential functions, exponential equations can be solved using logarithms. The base of the logarithm used can be one of the bases in the problem or a base available on the calculator (10 or e).

❶ Solve an exponential equation by using log with a base in the equation.

1. Take the log of each side using one of the bases in the problem.
2. Simplify the side with the same base as the one used for the logarithm. If the other side also has an exponent, use the power property.
3. Use the change of base property.
4. Use algebra to solve.

❶ Solve by using log with base 4.

a) $4^{5x} = 8$

1. $\log_4 4^{5x} = \log_4 8$
2. $5x = \log_4 8$
3. $5x = \frac{\log 8}{\log 4}$
4. $5x = 1.5$

$x = 0.3$

b) $4^{5x} = 8^{x+10}$

$\log_4 4^{5x} = \log_4 8^{x+10}$

$5x = (x + 10) \log_4 8$

$5x = (x + 10) \frac{\log 8}{\log 4}$

$5x = (x + 10)(1.5)$

$5x = 1.5x + 15$

$x \approx 4.29$

❷ Solve an exponential equation by using common log or natural log.

1. Take the common log or natural log of each side.
2. Use the power property.
3. Use the calculator to evaluate the logs.
4. Use algebra to solve.

❷ Solve by using log.

a) $4^{5x} = 8$

1. $\log 4^{5x} = \log 8$
2. $5x \log 4 = \log 8$
3. $5x(.60) \approx .90$
4. $3x \approx .90$

$x \approx .30$

b) $4^{5x} = 8^{x+10}$

$\log 4^{5x} = \log 8^{x+10}$

$5x \log 4 = (x + 10) \log 8$

$5x(.60) \approx (x + 10)(.90)$

$3.0x \approx 0.9x + 9$

$2.1x \approx 9$

$x \approx 4.29$

When both bases are powers of the same number (e.g., 8 and 16 are both powers of 2), exponential equations can be solved without a calculator.

❸ Solve an exponential equation without a calculator.

1. Find a number b that both bases are a power of.
2. Write each base as a power of b.
3. Apply the power of a power property of exponents: $(b^m)^n = b^{mn}$.
4. Apply $\log_b$ to each side.
5. Simplify by canceling the exponential and logarithmic functions.
6. Use algebra to solve.

❸ $4^{5x} = 8^{x+10}$

1. $b = 2$
2. $(2^2)^{5x} = (2^3)^{x+10}$
3. $2^{10x} = 2^{3x+30}$
4. $\log_2 2^{10x} = \log_2 2^{3x+30}$
5. $10x = 3x + 30$
6. $7x = 30$

$x = \frac{30}{7}$

5-E Exponential Situations

Many real-world problems can be solved by setting up and solving an equation of the form $f(x) = ab^x$.

❶ Translate a description of an exponential situation into an equation, and solve it.

1. Identify the value of the starting amount, if known. Label it a.
2. Identify the value of the ending amount, if known. It may be known only relative to the starting amount, such as $3a$ for three times as much or $0.8a$ for 20% less. Label it $f(x)$.
3. Identify the growth or decay factor, if known (see ❶ in 5-A). Label it b.
4. Identify the time period or number of times, if known. Label it x. Clarify the units, such as "years" or "years since 1990", as appropriate for the situation.
5. Solve the equation $f(x) = ab^x$. Logarithms will be used if solving for x.

❶ A radioactive substance is decaying at a rate of 10% per minute. In how long will only half of the original amount remain? (This duration is called its HALF LIFE.)

1. a is unknown.
2. $f(x) = \frac{a}{2}$
3. $b = 1 - .1 = 0.9$
4. x is unknown, and is measured in minutes.
5. $\frac{a}{2} = a(0.9)^x$

   $\frac{1}{2} = 0.9^x$

   $\log_{0.9} \frac{1}{2} = \log_{0.9} 0.9^x$

   $x = \frac{\log ½}{\log 0.9} \approx 6.6$ minutes

❶ It takes 6.6 minutes for half of the mass of a radioactive substance to decay. (That is, it has a half-life of 6.6 minutes.) What is its rate of decay?

1. a is unknown
2. $f(x) = \frac{a}{2}$
3. b is unknown
4. $x = 6.6$ minutes
5. $\frac{a}{2} = ab^{6.6}$

   $\frac{1}{2} = b^{6.6}$

   $b = \left(\frac{1}{2}\right)^{1/6.6} \approx 0.9$

   $r \approx 0.9 - 1 = -.1 = -10\%$

   The rate of decay is 10% per minute.

5-F Logarithmic Equations

To EXPONENTIATE an expression with base b is to make the expression an exponent of b. Exponentiation is used to solve logarithmic equations.

❶ Solve a logarithmic equation.

1. If the equation uses more than one base, use the change of base formula to get it to a single base.
2. Use basic algebra and properties of logarithms as needed to get a single log expression only on one side.
3. Exponentiate each side of the equation using the same base as the logarithm.
4. Simplify by canceling the exponential and logarithmic functions.
5. Solve. Disregard any solution that is extraneous due to requiring the log of a negative number.

❶ Solve.

a) $\log_x 400 = 5$

1.

2.

3. $x^{\log_x 400} = x^5$

4. $400 = x^5$

5. $x = 400^{1/5} \approx 3.31$

b) $8 \log_9 x - 2 \log_3 4x = 5$

$8 \frac{\log_3 x}{\log_3 9} - 2 \log_3 4x = 5$

$8 \frac{\log_3 x}{2} - 2 \log_3 4x = 5$

$2 \log_3 x - \log_3 4x = 2.5$

$\log_3 x^2 - \log_3 4x = 2.5$

$\log_3 \frac{x}{4} = 2.5$

$3^{\log_3 x/4} = 3^{2.5}$

$\frac{x}{4} = 3^{2.5}$

$x = 4(3^{2.5}) \approx 62.4$

c) $\log(x + 1) + \log(x - 2) = 1$

$\log(x^2 - x - 2) = 1$

$10^{\log(x^2 - x - 2)} = 10^1$

$x^2 - x - 2 = 10$

$x^2 - x - 12 = 0$

$(x - 4)(x + 3) = 0$

$x = 4$, $x \neq -3$

$x = -3$ is extraneous because it would involve taking the log of -2 and of -5 which are not defined