Chapter Four

Nonright Triangles

The Law of Sines

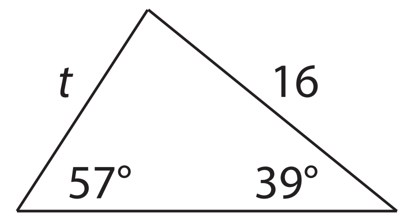
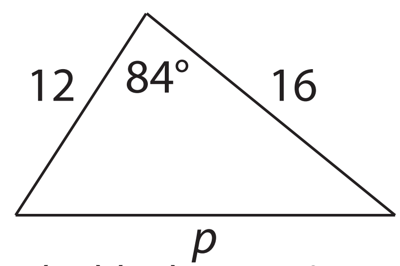
The Law of Cosines

Areas of Triangles

Solving Nonright Triangles

In nonright triangles, there is no hypotenuse, and each angle has two adjacent sides, making the original definitions of sine, cosine, and tangent (in terms of adjacent, opposite, and hypotenuse) meaningless.

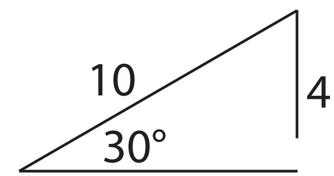
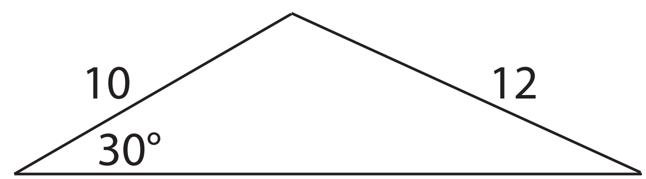
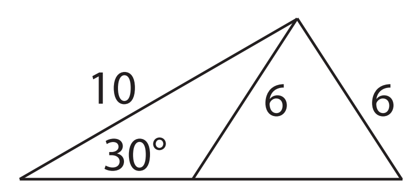
Instead, sides and angles in nonright triangles can be found with the **law of sines** or the **law of cosines**.

Method	When used	Equation	Example
Law of Sines	An angle and the side opposite it are known, and so is one other side or angle.	$\frac{a}{\sin A} = \frac{b}{\sin B}$	 $\frac{16}{\sin 57°} = \frac{t}{\sin 39°}$ $t = \left(\frac{16}{\sin 57°}\right) \sin 39° = 12.0$
Law of Cosines	Two sides are known, and so is the angle between them or the side between them.	$c^2 = a^2 + b^2 - 2ab \cos C$	 $p^2 = 12^2 + 16^2 - 2(12)(16) \cos 84°$ $p = \sqrt{144 + 256 - 384 \cos 84°} = 19.0$

The law of cosines can be algebraically rewritten as $c = \sqrt{a^2 + b^2 - 2ab \cos C}$ or as $C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$ to make it easier to calculate a side or an angle, respectively.

The Ambiguous Case of the Law of Sines

When solving for an angle using an inverse trig function, there are infinitely many solutions. For the law of sines, there may be zero, one, or two such angles that fit within the given triangle.

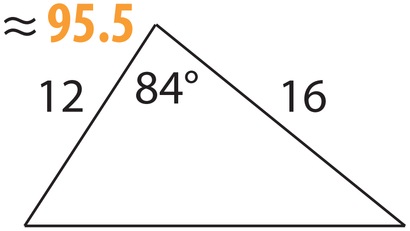
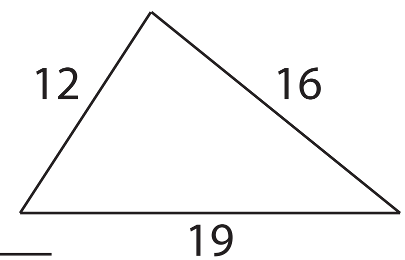
Solutions	When it happens	Example
zero	The side opposite the known angle is too short to connect the other two sides.	$A = 30°, b = 10, a = 4$ $\frac{\sin 30°}{4} = \frac{\sin B}{10}$ $\sin B = 10\left(\frac{\sin 30°}{4}\right) = 1.25$ (not possible) 
one	The side opposite the known angle is longer than the other known side.	$A = 30°, b = 10, a = 12$ $\frac{\sin 30°}{12} = \frac{\sin B}{10}$ $\sin B = 10\left(\frac{\sin 30°}{12}\right) \approx .42$ $B_1 \approx \sin^{-1} .42 \approx 25°$ $B_2 \approx 180° - 25° = 155°$ $C_2 \approx 180° - (30° + 155°) = -5°$ (not possible) 
two	The side opposite the known angle is shorter than the other known side, but still long enough to reach the unknown side.	$A = 30°, b = 10, a = 6$ $\frac{\sin 30°}{6} = \frac{\sin B}{10}$ $\sin B = 10\left(\frac{\sin 30°}{6}\right) \approx .83$ $B_1 \approx \sin^{-1} .83 \approx 56°$ $B_2 \approx 180° - 56° = 124°$ $C_2 \approx 180° - (30° + 124°) = 26°$ 

Areas of Triangles

The three **altitudes** of a triangle are the line segments from each vertex to the opposite side, connected at a right angle. Their lengths can be calculated for ΔABC as $h = b \sin C$, $h = c \sin A$, and $h = a \sin B$.

Since an altitude is a height and the side perpendicular to it is a base, the area of a triangle can be calculated using these two values.

If only the side lengths of a triangle are known, its area can be calculated by Hero's formula, which is based on the law of cosines and involves the semiperimeter $s = \frac{a+b+c}{2}$.

Method	Area	Example
½ base • height	$\frac{1}{2}ab \sin C$	area = $\frac{1}{2}(12)(16 \sin 84°) \approx$ **95.5** 
Hero's formula	$\sqrt{s(s-a)(s-b)(s-c)}$	 $s = \frac{12+16+19}{2} =$ **23.5** area = $\sqrt{23.5(11.5)(7.5)(4.5)} =$ **95.5**

If the angles and only one side are known, a second side can be found using the law of sines.