CHAPTER FOUR: NONRIGHT TRIANGLES

Review December 1 ❧ Test December 8

Chapter two solved right triangles using simple sine, cosine, and tangent equations. This chapter uses two new concepts, the law of sines and the law of cosines, to solve nonright triangles and find their areas. Some triangles requiring the use of $\sin^{-1}$ *do not actually exist, and if they do, a second triangle may or may not also exist. The possibility of a second triangle is because an angle* θ *calculated using* $\sin^{-1}$ *could also measure* $180° - \theta$*.*

4-A The Law of Sines — Tuesday • 11/22

law of sines • altitude • ambiguous case

1. Solve an AAS or ASA triangle.
2. Calculate an altitude of a triangle from a specific base.
3. Given SSA, identify whether zero, one, or two triangles exist.
4. Solve an SSA triangle.

4-B The Law of Cosines — Tuesday • 11/29

law of cosines

1. Solve an SSS or SAS triangle.

4-C Areas of Triangles — Thursday • 12/1

1. Find the area of a nonright triangle.

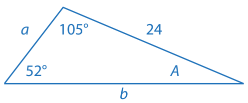
4-A The Law of Sines

The methods in 2-A of solving triangles only apply to right triangles. For nonright triangles, the law of sines or the law of cosines can be used.
The LAW OF SINES can be used to solve AAS, ASA, and SSA triangles: $\frac{a}{\sin A} = \frac{b}{\sin B}$.

❶ Solve an AAS or ASA triangle.

1. Substitute the known side and angles into the law of sines.
2. Solve for the unknown side by multiplying each side of the equation by the sine of the corresponding angle.
3. Subtract the two known angles from 180° to find the third angle.
4. Repeat steps 1 and 2, using the given side-angle pair, to find the third side.

❶



1. $\frac{b}{\sin 105°} = \frac{24}{\sin 52°}$
2. $b = \sin 105° \cdot \frac{24}{\sin 52°} \approx 29.4$
3. $A = 180° - 105° - 52° = 23°$
4. $\frac{a}{\sin 23°} = \frac{24}{\sin 52°}$

   $a = \sin 23° \cdot \frac{24}{\sin 52°} \approx 11.9$

The ALTITUDE of a Triangle is its height. Triangles have three altitudes, one from each base.

A triangle's altitude divides it into two right triangles if it is acute, or expands it into a right triangle if it is obtuse. In either case, its length can be calculated using right-triangle trig (see ❸ in 2-A). From base b, the altitude of triangle ABC is $h = c \sin A$ (or, equivalently, $h = a \sin C$).

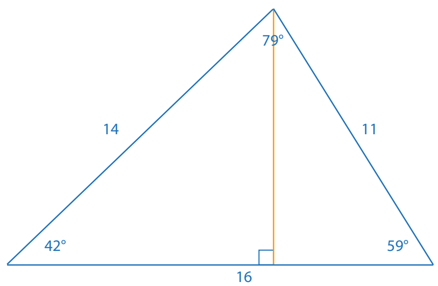
❷ Calculate an altitude of a triangle from a specific base.

1. Identify an angle between a known side and the base.
2. Multiply the side's length by the sine of the angle.

❷ For the triangle shown, calculate the altitude from each base.

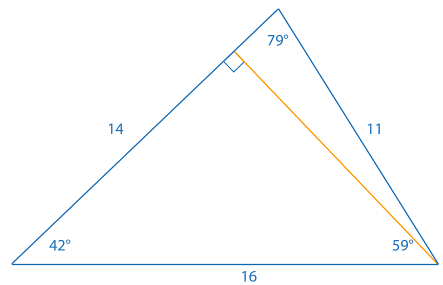
a) from base 16

2. $h = 14 \sin 42° \approx 9.4$



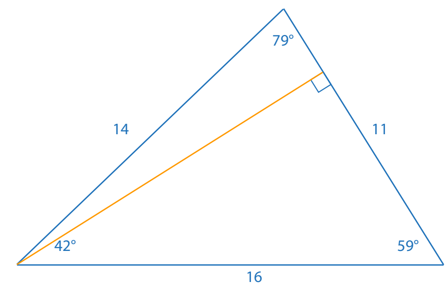
b) from base 14

$h = 11 \sin 79° \approx 10.8$



c) from base 11

$h = 16 \sin 59° \approx 13.7$



The AMBIGUOUS Case of the Law of Sines is when SSA information is given, because there can be zero, one, or two possible triangles.

❸ Given SSA, identify whether zero, one, or two triangles exist.

1. If $A \geq 90°$, compare a and c:
   If $a \leq c$, then no triangle exists.
   If $a > c$, then one triangle exists.
2. If $A < 90°$, calculate $h = c \sin A$ and compare a, c, and h:
   If $a < h$, then no triangle exists.
   If $a \geq c$ or $a = h$, then one triangle exists.
   If $h < a < c$, then two triangles exist.

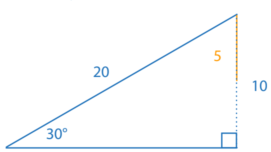
❸ State the number of triangles that exist for the given information.

a) $A = 30°$, $a = 5$, $c = 20$

2. $h = 20 \sin 30° = 10$

$10 > 5$

There are zero triangles because the side is too short to reach the base.

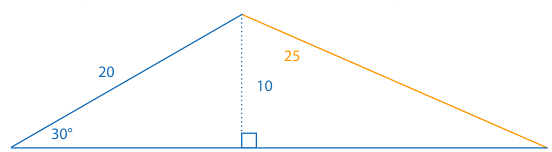


b) $A = 30°$, $a = 25$, $c = 20$

$h = 20 \sin 30° = 10$

$25 \geq 20$

There is one triangle because the side reaches the base in one direction.

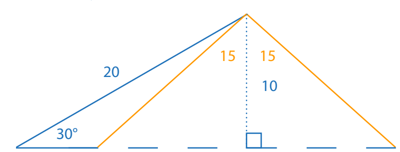


c) $A = 30°$, $a = 15$, $c = 20$

$h = 20 \sin 30° = 10$

$10 \leq 15 < 20$

There are two triangles because The side can reach the base in either direction.



❹ Solve an SSA triangle.

1. If the given angle is obtuse and the side opposite it is not the longest side, then there are zero triangles. Otherwise continue.
2. Write an equation using two of the fractions from the law of sines, but put the sides in the denominator. Label the angle with a subscript of 1 to indicate that it is part of the first triangle.
3. Solve for the sine.
4. If the sine is not within the range of sine ($-1 \le \sin\theta \le 1$) then there are zero triangles. Otherwise continue.
5. Use $\sin^{-1}$ to solve for the angle.
6. Subtract the two known angles from 180° to find the third angle.
7. Write an equation using two of the fractions, this time with the sides (one of them being the unknown side) in the numerator.
8. Solve.
9. Subtract the answer for step 5 from 180° to find a second angle that has the same sine. Label this supplementary angle with a subscript of 2 to indicate that it is part of the second triangle.
10. Using the answer from step 9 instead of step 5, repeat steps 6-8 to find the second triangle. If step 6 gives a negative angle, there is no second triangle.

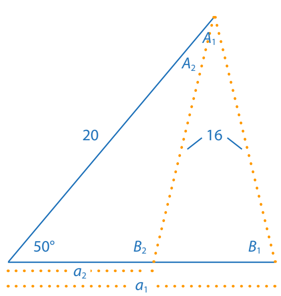
❹ Solve and sketch the following triangles, if they exist.

a) $b = 20, C = 50°, c = 16$

2. $\frac{\sin B_1}{20} = \frac{\sin 50°}{16}$
3. $\sin B_1 = 20 \bullet \frac{\sin 50°}{16} \approx .9576$
4. $-1 \le .9576 \le 1$, so there is a triangle.
5. $B_1 \approx \sin^{-1} .9576 \approx 73.3°$
6. $A_1 \approx 180° - 73.3° - 50° = 56.7°$
7. $\frac{a_1}{\sin 56.7°} \approx \frac{16}{\sin 50°}$
8. $a_1 \approx \sin 56.7° \bullet \frac{16}{\sin 50°} \approx 17.5$
9. $B_2 \approx 180° - 73.3° = 106.7°$
10. $A_2 \approx 180° - 106.7° - 50° = 23.3°$

$\frac{a_2}{\sin 23.3°} \approx \frac{16}{\sin 50°}$

$a_2 \approx \sin 23.3° \bullet \frac{16}{\sin 50°} \approx 8.3$



b) $b = 20, C = 50°, c = 25$

$\frac{\sin B_1}{20} = \frac{\sin 50°}{25}$

$\sin B_1 = 20 \bullet \frac{\sin 50°}{25} \approx .6128$

$-1 \le .6128 \le 1$, so there is a triangle

$B_1 \approx \sin^{-1} .6128 \approx 37.8°$

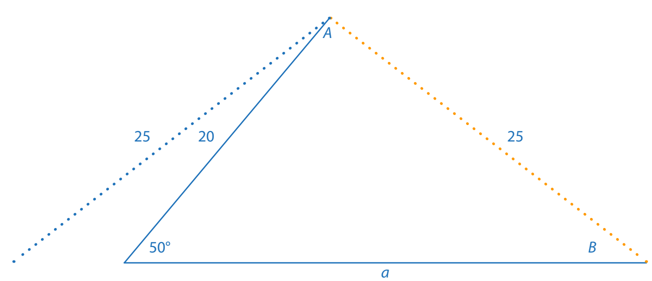
$A_1 \approx 180° - 37.8° - 50° = 92.2°$

$\frac{a_1}{\sin 56.7°} \approx \frac{25}{\sin 50°}$

$a_1 \approx \sin 92.2° \bullet \frac{25}{\sin 50°} \approx 32.6$

$B_2 \approx 180° - 37.8° = 142.2°$

$A_2 \approx 180° - 142.2° - 50° = -12.2° < 0°$, so there is no second triangle.

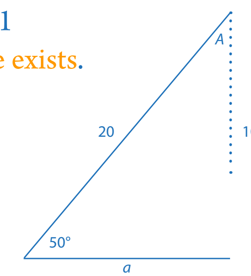


c) $b = 20, C = 50°, c = 10$

$\frac{\sin B_1}{20} = \frac{\sin 50°}{25}$

$\sin B_1 = 20 \bullet \frac{\sin 50°}{25} \approx 1.5321$

$1.5321 > 1$, so no triangle exists.



4-B The Law of Cosines

The LAW OF COSINES can be used to solve SSS and SAS triangles. It is commonly written $c^2 = a^2 + b^2 - 2ab\cos C$, but it is more useful when solved for a side c or an angle C:

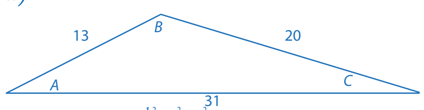
$c = \sqrt{a^2 + b^2 - 2ab\cos C}$ $C = \cos^{-1}\frac{a^2 + b^2 - c^2}{2ab}$

❶ Solve an SSS or SAS triangle.

1. For an SAS triangle, use the law of cosines formula above on the left to solve for a side.
   For an SSS triangle, use the law of cosines formula above on the right to solve for an angle.
2. Rewrite the law of cosines formula using the variables in the given triangle.
3. Plug in the three known values and calculate the unknown value.
4. To find a second angle, repeat steps 2 and 3 using the formula on the right, above, or use the law of sines.
5. Subtract the two known angles from 180° to find the third angle.

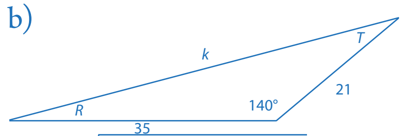
❶ Solve the triangles shown.

a)



1. $A = \cos^{-1}\frac{b^2 + c^2 - a^2}{2bc}$
2.
3. $A = \cos^{-1}\frac{31^2 + 13^2 - 20^2}{2(31)(13)} \approx 25.1°$
4. $B = \cos^{-1}\frac{a^2 + c^2 - b^2}{2ac}$
   $B = \cos^{-1}\frac{20^2 + 13^2 - 31^2}{2(20)(13)} \approx 138.9°$
5. $C \approx 180° - 25.1° - 138.9° = 16.0°$

b)



$c = \sqrt{a^2 + b^2 - 2ab\cos C}$
$k = \sqrt{r^2 + t^2 - 2rt\cos K}$
$k = \sqrt{21^2 + 35^2 - 2(21)(35)\cos 140°} \approx 52.8$
$R = \cos^{-1}\frac{k^2 + t^2 - r^2}{2kt} \approx 14.9°$
$R \approx \cos^{-1}\frac{35^2 + 52.8^2 - 21^2}{2(25)(52.8)} \approx 14.9°$
$T \approx 180° - 140° - 14.9° = 25.1°$

4-C Areas of Triangles

The area of a triangle is area = ½ bh, where b is the base and h is the altitude. Since $h = c \sin A$, area = ½ $bc \sin A$.

The area of a triangle can also be expressed by Heron's formula: area = $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$ is half the perimeter.

❶ Find the area of a nonright triangle.

1. If only the sides are known, calculate the semiperimeter $s = \frac{1}{2}(a+b+c)$ and use it to calcualate area = $\sqrt{s(s-a)(s-b)(s-c)}$.
2. If an angle is known between two known sides, multiply the sine of this angle by ½ the product of the sides: area = ½ $bc \sin A$.
3. If two angles and one side are known, use the law of sines to find one of the other sides, and then do step 2, above.

❶ Find the area of a triangle with $m = 8$, $n = 13$, and $p = 15$.

1. $s = \frac{1}{2}(8 + 13 + 15) = 18$

   area = $\sqrt{18(18-8)(18-13)(18-15)} \approx 52.0$

❶ Find the area of a triangle with $m = 20$, $n = 14$, and $P = 116°$.

2. area = ½ • 20 • 14 sin 116° ≈ 125.8

❶ Find the area of a triangle with $m = 24$, $M = 99°$, $N = 31°$.

3. $\frac{n}{\sin 31°} = \frac{24}{\sin 99°}$

   $n = \sin 31° \cdot \frac{24}{\sin 99°} \approx 12.5$

   $A = 180° - 99° - 31° = 50°$

2. area ≈ ½ • 24 • 12.5 sin 50° ≈ 114.9