Chapter Three

Trigonometric Equations

Solving Simple Trigonometric Equations Algebraically

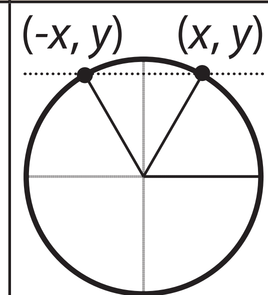
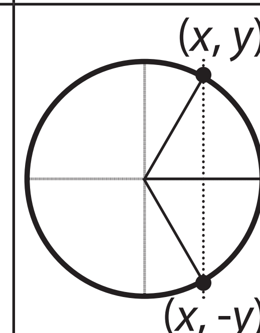
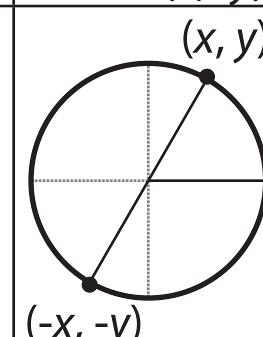
Solving Complicated Trigonometric Equations Algebraically

Graphs of Sine and Cosine Functions

Solving Trigonometric Equations Graphically

Two Solutions to Simple Trigonometric Equations

Based on the symmetry of the unit circle, simple trig equations such as sin $A = ½$ have two solutions between 0° and 360°. If not labeled on the unit circle, these can be found by using an inverse function and then finding a second solution as shown below.

Function	1st Solution	2nd Solution	Reason	Sketch
sine	$A_1 = \sin^{-1} b$	$A_2 = 180° - A_1$	Subtracting from 180° reflects the terminal side across the *y*-axis, keeping the same *y*-coordinate.	
cosine	$A_1 = \cos^{-1} b$	$A_2 = 360° - A_1$	Subtracting from 360° reflects the terminal side across the *x*-axis, keeping the same *x*-coordinate.	
tangent	$A_1 = \tan^{-1} b$	$A_2 = 180° + A_1$	Adding 180° rotates the terminal side to the opposite quadrant, keeping the same *x*-coordinate and *y*-coordinate. (The sign of each coordinate switches, but since they both switch, this has no effect.)	

Solving Trigonometric Equations

Solving more complicated trigonometric equations involves algebra and trigonometric identities.

Step	$5\sec^2 3x = 10$	Notes for example
Isolate the trig function.	$\sec^2 3x = 2$	Divide each side by 5.
If cot x, sec x, or csc x is isolated, take the reciprocal.	$\cos^2 3x = ½$	The reciprocal of sec x is cos x. (The argument is unchanged.)
If the trig function is squared, take the sqaure root.	$\cos 3x = \pm\sqrt{½} = \pm\frac{\sqrt{2}}{2}$	Don't forget to put $\pm$. $\sqrt{½}$ can be simplified to $\frac{\sqrt{2}}{2}$ and.
Apply the inverse trig function.	$\cos^{-1}\cos 3x = \cos^{-1}\frac{\sqrt{2}}{2} = 45°$ $\cos^{-1}\cos 3x = \cos^{-1}-\frac{\sqrt{2}}{2} = 135°$	Make sure to do both if there is a $\pm$.
Find a second solution for each solution found.	$3x = 360° - 45° = 315°$ $3x = 360° - 135° = 225°$	Subtract the $\cos^{-1}$ results from 360° (see previous slide).
Add 360°n (or $2\pi n$) to each solution.	$3x = 45° + 360°n$ $3x = 135° + 360°n$ $3x = 225° + 360°n$ $3x = 315° + 360°n$	This finds coterminal solutions.
Use algebra to solve.	$x = 15° + 120°n$ $x = 45° + 120°n$ $x = 75° + 120°n$ $x = 105° + 120°n$	Divide each side by 3 to solve for x. Don't forget to divide the 360°n.

Solving Trigonometric Equations by Factoring

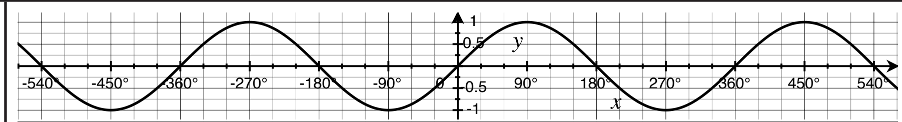
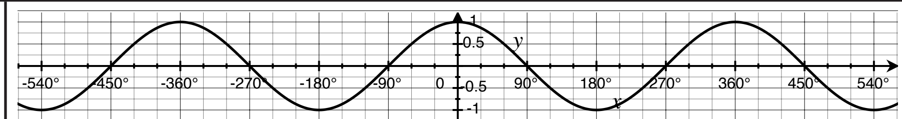
In many cases, trigonometric equations can be solved most easily by factoring, like polynomial equations.

Step	$x^3 + x^2 = 20x$	$\tan^3 x + \tan^2 x = 20 \tan x$
Set the equation equal to 0.	$x^3 + x^2 - 20x = 0$	$\tan^3 x + \tan^2 x - 20 \tan x = 0$
Factor the expression.	$x(x^2 + x - 20) = 0$ $x(x - 4)(x + 5) = 0$	$(\tan x)(\tan^2 x + \tan x - 20) = 0$ $(\tan x)(\tan x - 4)(\tan x + 5) = 0$
Set each factor equal to zero.	$x = 0$ $x - 4 = 0$ $x + 5 = 0$	$\tan x = 0$ $\tan x - 4 = 0$ $\tan x + 5 = 0$
Solve each equation.	$x = 0$ $x = 4$ $x = -5$	$x = \tan^{-1} 0 = 0°$ $x = \tan^{-1} 4 \approx 76.0°$ $x = \tan^{-1} -5 \approx 78.7°$
Find additional solutions.*	n/a	$x = 180°n$ $x \approx 76.0° + 180°n$ $x \approx 78.7° + 180°n$

* For tangent (which is most common in trig factoring problems), additional solutions can be found by simply adding $180°n$ to each original solution.

Graphs of Sine and Cosine

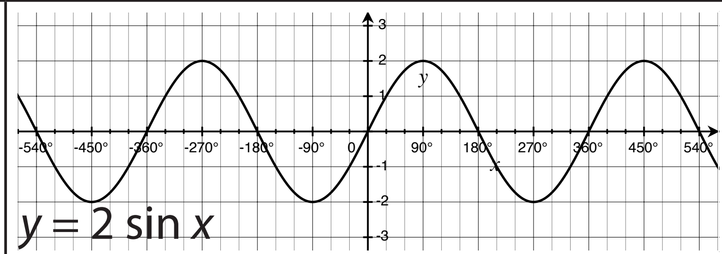
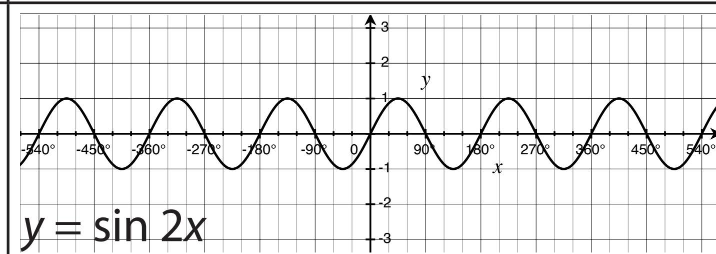
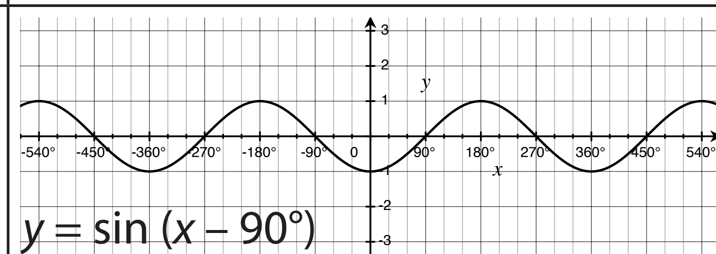
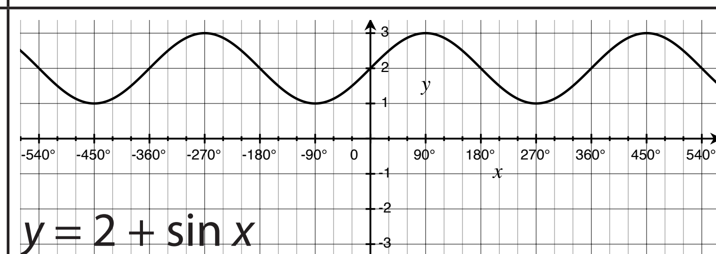
The graph of $y = \sin x$ is a wave that, because of coterminal angles, repeats itself every 360°. The same is true for $y = \cos x$.

Function	At *y*-axis	Graph
$y = \sin x$	at middle of wave, going up	
$y = \cos x$	at top of wave, going down	

The graph of $y = \cos x$ is the same as the graph of $y = \sin x$, except that it is translated to the left by 90°: $\cos x = \sin (x + 90°)$. Likewise, $\sin x = \cos (x - 90°)$.

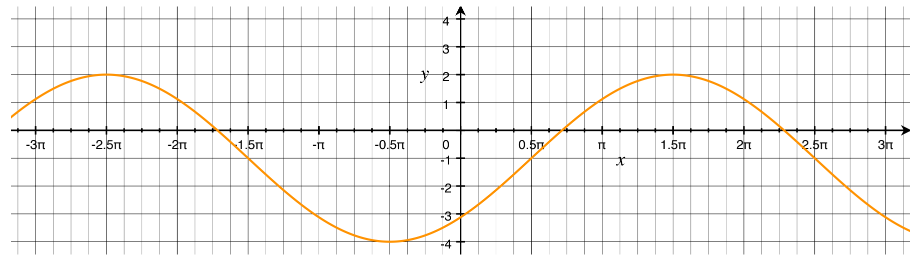
Transformations of Sine and Cosine

Like any function, sine and cosine can be transformed by translation, stretch, and reflection (see 1-E). For translations and stretches, there is terminology specific to trig functions.

Aspect	Equation	Transformation	Graphic example
Amplitude	$y = a \sin x$	vertical stretch of a	
Period	$y = \sin bx$	horizontal stretch of $^1/_b$	
Phase shift	$y = \sin (x - c)$	horizontal translation of c	
Vertical shift	$y = d + \sin x$	vertical translation of d	

Equations of Sine and Cosine Graphs

The equation of a sine graph $y = d + a \sin b(x - c)$ can be deterimined by identifying the parameters a, b, c, and d, based on the amplitude, period, horizontal shift, and vertical shift of the graph.

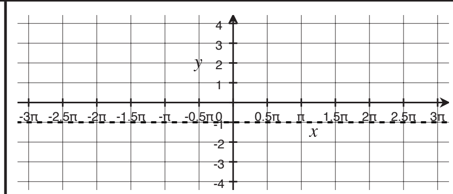
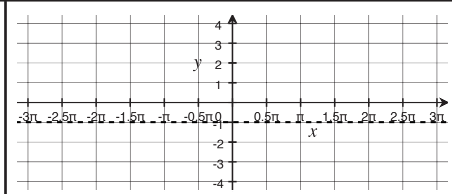
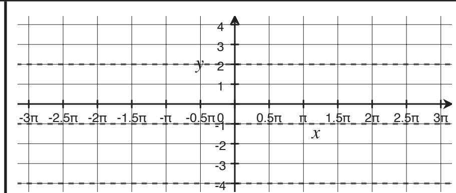
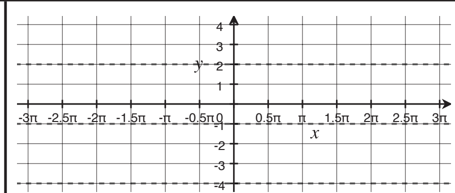
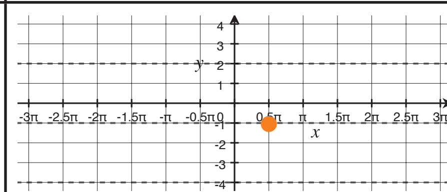
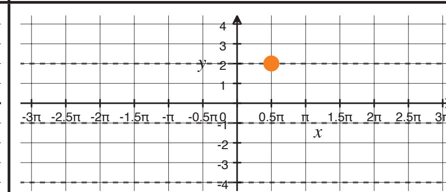
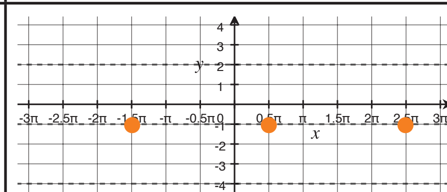
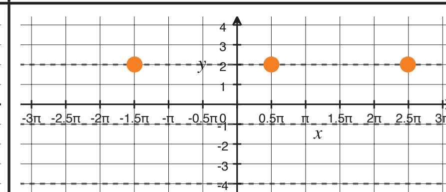


The equation of the graph above can be written $y = -1 + 3 \sin \frac{1}{2}(x - \frac{\pi}{2})$ or $y = -1 + 3 \cos \frac{1}{2}(x - \frac{3\pi}{2})$.

Parameter	How to identify	Sine example	Cosine Example
d	average of the the highest and lowest y value	$d = -1$	same
a	distance from d to the top of the curve	$a = 2 - (-1) = 3$	same
b	$b = \frac{2\pi}{\text{period}}$ (or $b = \frac{360°}{\text{period}}$), where the period is how far the graph goes before repeating, such as from one peak to the next	period = $1.5\pi - (-2.5\pi) = 4\pi$ $b = \frac{2\pi}{4\pi} = \frac{1}{2}$	same
c	the distance from the y-axis to where the graph crosses the line $y = d$ going up (for sine) or where the graph peaks (for cosine)	At $x = c = .5\pi$, $y = d$ and the graph is going upward.	The graph peaks at $x = c = 1.5\pi$.

Sketches of Sine and Cosine graphs

A sine graph or cosine graph can be sketched by identifying *a*, *b*, *c*, and *d*, and applying them as shown below.

Aspect	Procedure	$y = -1 + 3\sin\left(x - \frac{\pi}{2}\right)$	$y = -1 + 3\cos\left(x - \frac{\pi}{2}\right)$
vertical shift	The middle of the curve is at $y = d$.		
amplitude	The top of the curve is at $y = d + a$, and the bottom is at $y = d - a$.		
phase shift	For sine, plot a point at (c, d), or for cosine, plot a point $(c, d + a)$.		
period	The period is $\frac{2\pi}{b}$. Plot a point this far to the right of the first point plotted, and another point this far to the left.		
curve	Sketch a sine or cosine curve from each point to the next. If *a* is negative, sketch it upside-down.