CHAPTER THREE: TRIGONOMETRIC EQUATIONS — Review November 8 ↝ Test November 17

Trigonometric equations can be solved graphically or algebraically. Solving algebraically involves using inverse trigonometric functions, and may also require the use of trigonometric identities. Because of coterminal angles, trigonometric equations have infinitely many solutions. In addition, they have noncoterminal solutions, because there are two angles with the same sine (or cosine or tangent) in a circle.

3-A Solving Simple Trigonometric Equations Algebraically — Thursday • 10/27

general solution

❶ Use the unit circle to find two solutions to a simple trigonometric equation.
❷ Use an inverse function to find two solutions to a simple trigonometric equation.
❸ Find all solutions to a simple trigonometric equation within a given range.
❹ Find the general solution to a simple trigonometric equation.

3-B Solving Complicated Trigonometric Equations Algebraically — Monday • 10/31

❶ Use algebra and basic trig identities to simplify an equation, and solve it.
❷ Solve a trigonometric equation by factoring.

3-C Graphs of Sine and Cosine Functions — Thursday • 11/3

amplitude • period • phase shift

❶ Stretch $y = \sin x$ to have a specified amplitude and period.
❷ Translate $y = a \sin bx$ with a phase shift of c to the right and a vertical shift of d upward.
❸ Identify the amplitude, period, phase shift, and vertical shift of an equation of the form $y = d + a \sin b(x - c)$.
❹ Write the equation of a graphed sine or cosine function.
❺ Sketch $y = d + a \sin b(x - c)$ or $y = d + a \cos b(x - c)$.
❻ Sketch $y = d + a \sin (bx - bc)$ or $y = d + a \cos (bx - bc)$.

3-D Solving Trigonometric Equations Graphically — Tuesday • 11/8

❶ Solve a system of two equations by graphically finding the points of intersection.
❷ Solve an equation by finding the points of intersections of two graphs.

3-A Solving Simple Trigonometric Equations Algebraically

Except for sin $\theta = 1$, sin $\theta = -1$, cos $\theta = 1$, and cos $\theta = -1$, every simple trigonometric equation has two solutions on the unit circle.
For sine, the two solutions have the same y and thus are horizontal reflections of each other.
For cosine, the two solutions have the same x and thus are vertical reflections of each other.
For tangent, the two solutions have the same x and the same y except with opposite signs, and thus they are 180° rotations of each other.

❶ Use the unit circle to find two solutions to a simple trigonometric equation.

1. For sine, find the two points on the unit circle with the given value as y.
   For cosine, find the two points on the unit circle with the given value as x.
   For tangent, find the two points on the unit circle with the given value as $\frac{y}{x}$.
2. The solutions are the angles for the points found in step 1.

❶ a) $\sin\theta = \frac{1}{2}$
1. $(\frac{\sqrt{3}}{2}, \frac{1}{2})$
   $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$
2. $\theta = 30°$
   $\theta = 150°$

b) $\cos\theta = \frac{\sqrt{3}}{2}$
$(\frac{\sqrt{3}}{2}, \frac{1}{2})$
$(\frac{\sqrt{3}}{2}, -\frac{1}{2})$
$\theta = 30°$
$\theta = 330°$

c) $\tan\theta = \frac{\sqrt{3}}{3}$
$(\frac{\sqrt{3}}{2}, \frac{1}{2})$
$(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$
$\theta = 30°$
$\theta = 210°$

❷ Use an inverse function to find two noncoterminal solutions to a simple trigonometric equation.

1. Apply $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ to each side to find the first solution θ.
2. For sine, a second solution is $180° - \theta$.
   For cosine, a second solution is $360° - \theta$.
   For tangent, a second solution is $180° + \theta$.

❷ a) $\sin\theta = \frac{1}{2}$
1. $\theta = \sin^{-1}\frac{1}{2} = 30°$
2. $\theta = 180° - 30° = 150°$

b) $\cos\theta = \frac{\sqrt{3}}{2}$
$\theta = \cos^{-1}\frac{\sqrt{3}}{2} = 30°$
$\theta = 360° - 30° = 330°$

c) $\tan\theta = \frac{\sqrt{3}}{2}$
$\theta = \tan^{-1}\frac{\sqrt{3}}{3} = 30°$
$\theta = 180° + 30° = 210°$

Because coterminal angles have the same sine, cosine, and tangent as each other, trigonometric equations have infinitely many solutions.

❸ Find all solutions to a simple trigonometric equation within a given range.

1. Find two noncoterminal solutions (see ❶ or ❷).
2. List the first solution and several coterminal angles above and below it. Highlight the ones that are in the given range.
3. Repeat step 2 for the second solution.
4. List all the solutions in order.

❸ Find all solutions to $\sin\theta = \frac{1}{2}$ in the range $-360° < \theta < 810°$.

1. $\theta = 30°, 150°$
2. …, -690°, -330°, 30°, 390°, 750°, 1110°, 1470°, …
3. …, -570°, -210°, 150°, 510°, 860°, 1220°, 1580°, …
4. $\theta = -330°, -210°, 30°, 150°, 390°, 510°, 750°$

The GENERAL Solution to a trigonometric equation is the set of all solutions. It can be expressed by including "+ $2\pi n$" (or + "$360°n$") with each of the two original solutions, where n represents any integer. This shows that each solution remains true after adding or subtracting any number of complete circles.

Because tangent represents a reflection in both directions, it is simpler to find only one solution and include "+ πn" (or + "$180°n$") instead.

❹ Find the general solution to a simple trigonometric equation.

1. Find two noncoterminal solutions (see ❶ or ❷).
2. If using radians, include "+ $2\pi n$" for each solution.
   If using degrees, include "+ $360n°$" for each solution.

❹ a) $\sin\theta = \frac{1}{2}$

1. $\theta = 30°, 150°$ (see ❸).
2. $\theta = 30° + 360°n$
   $\theta = 150° + 360°n$

b) $\tan\theta = 1.38$

$\theta = \tan^{-1} 1.38 \approx 54°$

$\theta \approx 54° + 180°n$ (this is a simplifed version of "$\theta = 54° + 360n°$, $\theta = 234° + 360°n$")

3-B Solving Complicated Trigonometric Equations Algebraically

More complicated trigonometric equations can be solved using the methods of 3-A if they are first simplified using algebra and trigonometric identities.

❶ Use algebra and basic trig identities to simplify an equation, and solve it.

1. If $\sin\theta$ is a factor of one side of the equation and $\cos\theta$ is a factor of the other side, divide each side by $\cos\theta$ and simplify $\frac{\sin\theta}{\cos\theta}$ to $\tan\theta$.
2. Use algebra to isolate the trig function on one side of the equation.
3. If $\csc\theta$, $\sec\theta$, or $\cot\theta$ is isolated on one side of the equation, change it to $\sin\theta$, $\cos\theta$, or $\tan\theta$ by taking the reciprocal of each side of the equation.
4. If the isolated trig function is squared, take the square root of each side of the equation. Remember to put ±.
5. Apply $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ to each side to solve for the argument of the sine, cosine, or tangent function.
6. Find a second solution (see ❷ in 3-A) for each solution found in step 5.
7. Add $2\pi n$ (or $360°n$) to each value.
8. Use algebra to solve.
9. If a range is given, see ❸ in 3-A to find all specific solutions that are within the range, and list the solutions in order.

❶ Find all solutions to $5\sec^2 3\theta = 10$ in the range $0° < \theta < 360°$.

2. $\sec^2 3\theta = 2$
3. $\cos^2 3\theta = \frac{1}{2}$
4. $\cos 3\theta = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$
5. $3\theta = \cos^{-1}\frac{\sqrt{2}}{2} = 45°$
   $3\theta = \cos^{-1}-\frac{\sqrt{2}}{2} = 135°$
6. $3\theta = 360° - 45° = 315°$
   $3\theta = 360° - 135° = 225°$
7. $3\theta = 45° + 360°n$
   $3\theta = 135° + 360°n$
   $3\theta = 315° + 360°n$
   $3\theta = 225° + 360°n$
8. $\theta = 15° + 120°n$
   $\theta = 45° + 120°n$
   $\theta = 105° + 120°n$
   $\theta = 75° + 120°n$
9. 15°, 135°, 255°, 375°, 495°, …
   45°, 165°, 285°, 405°, 525°, …
   105°, 225°, 345°, 465°, 585°, …
   75°, 195°, 315°, 435°, 555°, …
   $\theta = 15°, 45°, 75°, 105°, 135°, 165°, 195°, 225°, 255°, 285°, 315°, 345°$

❷ Solve a trigonometric equation by factoring.

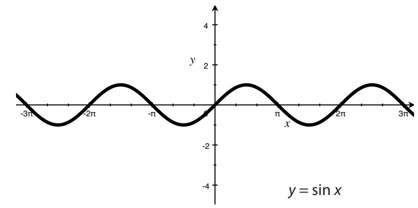
1. Set the equation equal to zero.
2. Factor the expression (see 2-E).
3. Set the first factor to zero and solve for the general solution (see 3-B).
4. Repeat step 3 for each additional factor.
5. If a range is given, see ❸ in 3-A to find all specific solutions that are within the range, and list the solutions in order.

❷ Find all solutions to $\tan^3 3\theta + \tan^2 3\theta = 20 \tan 3\theta$ in the range $0° \le \theta \le 90°$.

1. $\tan^3 3\theta + \tan^2 3\theta - 20 \tan 3\theta = 0$
2. $(\tan 3\theta)\ (\tan^2 3\theta + \tan 3\theta - 20) = 0$
   $(\tan 3\theta)\ (\tan 3\theta - 4)\ (\tan 3\theta + 5) = 0$
3. $\tan 3\theta = 0$
   $\tan^{-1} \tan 3\theta = \tan^{-1} 0$
   $3\theta = 0° + 180°n$
   $\theta = 60°n$
4. $\tan 3\theta - 4 = 0$
   $\tan 3\theta = 4$
   $\tan^{-1} \tan 3\theta = \tan^{-1} 4$
   $3\theta \approx 76.0° + 180°n$
   $\theta \approx 25° + 60°n$
   $\tan 3\theta + 5 = 0$
   $\tan 3\theta = -5$
   $\tan^{-1} \tan 3\theta = \tan^{-1} -5$
   $3\theta \approx 78.7° + 180°n$
   $\theta \approx 26° + 60°n$
5. 0°, 60°, 120°, …
   25°, 85°, 145°, …
   26°, 86°, 145°, …
   $\theta \approx$ 0°, 25°, 26°, 60°, 85°, 86°

3-C Graphs of Sine and Cosine Functions

The graph of $y = \sin x$ is shown at right.



It and other trig functions can be translated, stretched, and reflected. The procedure is the same as for other functions (see 1-E), but there is terminology specific to trig functions.

The AMPLITUDE of a sine function is the distance from the middle to the peaks, that is, half the total height.

The graph of $y = a \sin x$ is a vertical stretch of $y = \sin x$. Amplitude = $|a|$, and if a is negative there is a vertical reflection.

The PERIOD of a sine function is the horizontal distance for the graph to repeat itself.

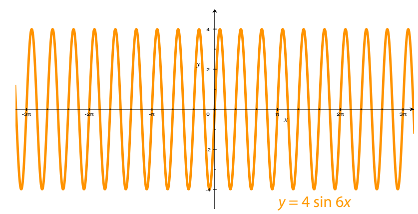
The graph of $y = \sin bx$ is a horizontal stretch of $y = \sin x$. Period = $\frac{2\pi}{b}$ (or $\frac{360°}{b}$). Likewise, $b = \frac{2\pi}{\text{period}}$ (or $\frac{360°}{\text{period}}$).

❶ Stretch $y = \sin x$ to have a specified amplitude and period.

1. Calculate $b = \frac{2\pi}{\text{period}}$ (or $\frac{360°}{\text{period}}$).
2. Use b as the coefficient for x.
3. Multiply the whole expression by the amplitude.

❶ Write a sine equation that has an amplitude of 4 and a period of $\frac{\pi}{3}$.

1. $b = \frac{2\pi}{\pi/3} = 2\pi \bullet \frac{3}{\pi} = 6$
2. $y = \sin 6x$
3. $y = 4 \sin 6x$



The vertical translation of a sine function is the distance the graph is moved upward.

The graph of $y = d + \sin x$ is a vertical translation of $y = \sin x$. Vertical translation = d.

The PHASE SHIFT of a sine function is the distance the graph is moved to the right.

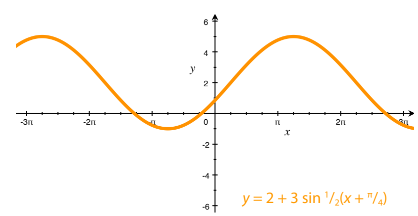
The graph of $y = \sin(x - c)$ is a horizontal translation of $y = \sin x$. Phase shift = c.

❷ Translate $y = a \sin bx$ with a phase shift of c to the right and a vertical shift of d upward.

1. Identify c, the amount it is translated to the right.
2. Identify d, the amount it is translated up.
3. The image is $y = d + a \sin b(x - c)$.

❷ Translate $y = 3 \sin 2x$ left by $\frac{\pi}{4}$ and up by 2.

1. $c = -\frac{\pi}{4}$
2. $d = 2$
3. $y = 2 + 3 \sin 2\left(x + \frac{\pi}{4}\right)$

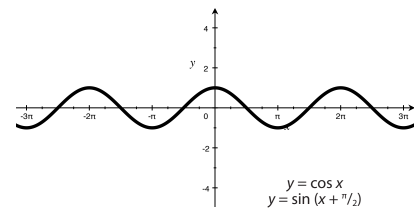


❸ Identify the amplitude, period, phase shift, and vertical shift of an equation of the form $y = d + a \sin b(x - c)$.

1. Identify a, b, c, and d.
2. The amplitude is $|a|$.
3. The period is $\frac{2\pi}{b}$ (or $\frac{360°}{b}$).
4. The phase shift is c.
5. The vertical shift is d.

❸ $y = 4 - 5 \sin 6(x + \frac{\pi}{8})$

1. $a = -5$ 2. amplitude = $|-5| = 5$
   $b = 6$ 3. period = $\frac{2\pi}{6} = \frac{\pi}{3}$
   $c = -\frac{\pi}{8}$ 4. phase shift = $-\frac{\pi}{8}$
   $d = 4$ 5. vertical shift = 4



The graph of $y = \cos x$ is the same as that of $y = \sin x$ except with a phase shift of $-\frac{\pi}{2}$ (-90°) as shown. Any sine function can be rewritten as a cosine function by subtracting $\frac{\pi}{2}$ (90°) from x in the equation.

$\cos(x - \frac{\pi}{2}) = \sin x$

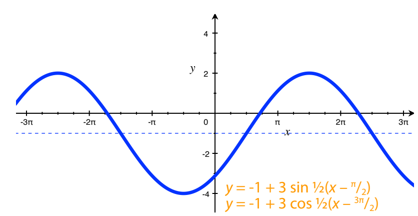
$\sin(x + \frac{\pi}{2}) = \cos x$

Sine waves and cosine waves are called SINUSOIDS. The horizontal line through the middle of a sinusoid is called the SINUSOIDAL AXIS.

❹ Write the equation of a graphed sine or cosine function.

1. Sketch the sinusoidal axis.
2. Identify the vertical shift d by noting the position of the sinusoidal axis.
3. Identify the amplitude a by finding the distance from the sinusoidal axis to the top or bottom of the graph.
4. Identify the period by noting how far the graph goes horizontally before repeating itself.
5. Calculate $b = \frac{2\pi}{\text{period}}$ (or $\frac{360°}{\text{period}}$).
6. For sine, identify the phase shift c as how far to the right of the origin the graph passes upward through the midline.
   For cosine, identify the phase shift c as how far to the right of the origin the graph hits a peak.
7. The equation is $y = d + a \sin b(x - c)$ or $y = d + a \cos b(x - c)$.

❹ Write a sine equation and a cosine equation for the graph at right.

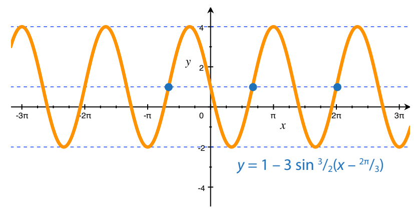


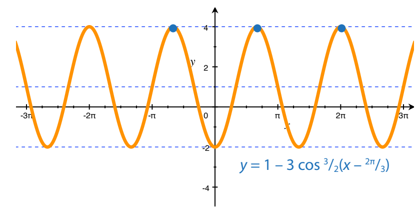
2. $d = -1$
3. $a = 3$
4. period = 4π
5. $b = \frac{2\pi}{4\pi} = \frac{1}{2}$
6. for sine, $c = \frac{\pi}{2}$ for cosine, $c = \frac{3\pi}{2}$
7. $y = -1 + 3 \sin \frac{1}{2}(x - \frac{\pi}{2})$ $y = -1 + 3 \cos \frac{1}{2}(x - \frac{3\pi}{2})$

❺ Sketch $y = d + a \sin b(x - c)$ or $y = d + a \cos b(x - c)$.

1. Identify a, b, c, and d.
2. Using dotted lines, sketch the midline $y = d$, the top of the wave $y = d + a$, and the bottom of the wave $y = d - a$.
3. Plot a point at (c, d) for sine or $(c, d + a)$ for cosine.
4. Calculate period = $\frac{2\pi}{b}$ (or $\frac{360°}{b}$).
5. Plot a point one period after the first point and another one period before it.
6. Draw a sine or cosine curve between the top and bottom line from each point to the next. If a is negative, reflect the curve vertically.
7. Continue the curve in each direction.
8. Draw a solid line for the axis $x = 0$ and the axis $y = 0$.

❺ Graph $y = 1 - 3 \sin \frac{3}{2}(x - \frac{2\pi}{3})$ and $y = 1 - 3 \cos \frac{3}{2}(x - \frac{2\pi}{3})$.





1. $a = -3$, $b = \frac{3}{2}$, $c = \frac{2\pi}{3}$, $d = 1$
2. top: $y = 1 + 3 = 4$
   midline: $y = 1$
   bottom: $y = 1 - 3 = -2$
3. $(\frac{2\pi}{3}, 1)$ for sine
   $(\frac{2\pi}{3}, 4)$ for cosine
4. period = $\frac{2\pi}{3/2} = 2\pi \bullet \frac{2}{3} = \frac{4\pi}{3}$
5. $\frac{2\pi}{3} + \frac{4\pi}{3} = 2\pi$
   $\frac{2\pi}{3} - \frac{4\pi}{3} = -\frac{2\pi}{3}$

❻ Sketch $y = d + a \sin(bx - bc)$ or $y = d + a \cos(bx - bc)$.

1. Factor out the b to rewrite the equation as $y = d + a \sin b(x - c)$ or $y = d + a \cos b(x - c)$.
2. Follow the directions for ❺.

❻ Graph $y = 1 - 3 \sin(\frac{3x}{2} - \pi)$.

1. $y = 1 - 3 \sin \frac{3}{2}(x - \frac{2\pi}{3})$
2. See ❺.

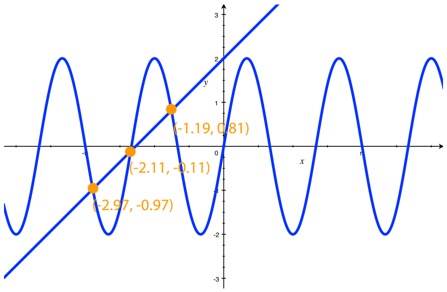
3-D Solving Trigonometric Equations Graphically

The calculator can give the exact points of intersection of two graphs, and thus the solution to a system of equations.
On the calculator, $\sin^2 x$ is typed `sin(X)`^{2}.

❶ Solve a system of two equations by graphically finding the points of intersection.

1. Graph each equation simultaneously, using the correct mode (radian or degree) for the problem.
2. If the points of intersection you want to find are off the screen, change the viewing area. In this chapter, use `ZTrig` for the zoom.
3. Push [CALC] and choose `intersect`.
4. Push [ENTER] for `First curve?` and again for `Second curve?`.
5. For `Guess?` use the left or right arrow key to move the cursor to a point of intersection and push [ENTER].
6. Repeat steps 2-5 for each additional point of intersection.

❶ Find the points of intersection of $f(x) = 2 \sin 3x$ and $g(x) = x + 1$.



1. Y_1=2sin3X
   Y_2=X+1
5. (-2.97, -0.97)
6. (-2.11, -0.11)
   (-1.19, 0.81)

❷ Solve an equation by finding the points of intersections of two graphs.

1. For Y_1, enter the expression on the left side of the equation.
2. For Y_2, enter the expression on the right side of the equation.
3. Solve this system (see ❶). Note that the solutions are the x-values, not the (x, y) coordinates, because there is no y in the equation.

❷ $2 \sin 3x = x + 1$

1. Y_1=2sin3X
2. Y_2=X+1
3. $x \approx -2.97, -2.11, -1.19$ (see ❶)