Chapter Two

Trigonometric Functions

Angles in Right Triangles

Angles in Circles

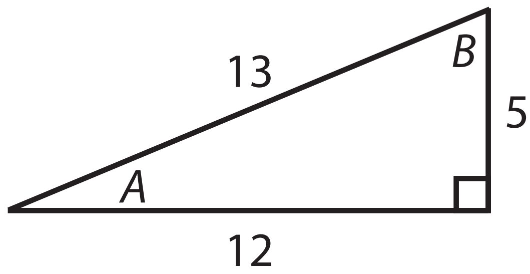
Radians

The Unit Circle

Trigonometric Identities

Trigonometric Functions in Right Triangles

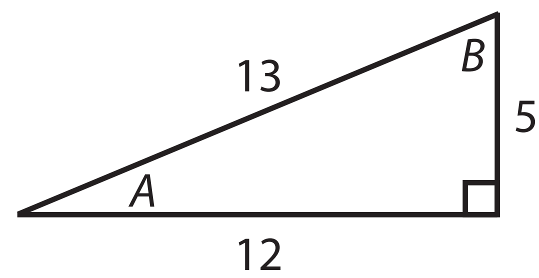
There are six possible ratios that can be made using the lengths of two of the sides of a right triangle, and each is a trigonometric function of the angle. For example, sine is the length of the side opposite the angle divided by the length of the hypotenuse. If the other angle is used, the opposite side becomes the adjacent side, and vice versa.



Function	Ratio	Reciprocal	Example *A*	Example *B*
sine	$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$	cosecant	$\sin A = \frac{5}{13}$	$\sin B = \frac{12}{13}$
cosine	$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$	secant	$\cos A = \frac{12}{13}$	$\cos B = \frac{5}{13}$
tangent	$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$	cotangent	$\tan A = \frac{5}{12}$	$\tan B = \frac{12}{5}$
cotangent	$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$	tangent	$\cot A = \frac{12}{5}$	$\cot B = \frac{5}{12}$
secant	$\sec\theta = \frac{\text{hypotenuse}}{\text{adjecent}}$	cosine	$\sec A = \frac{13}{12}$	$\sec B = \frac{13}{5}$
cosecant	$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$	sine	$\csc A = \frac{13}{5}$	$\csc B = \frac{13}{12}$

Inverse Trigonometric Functions

Trigonometric functions find the ratio of sides made by a given angle.



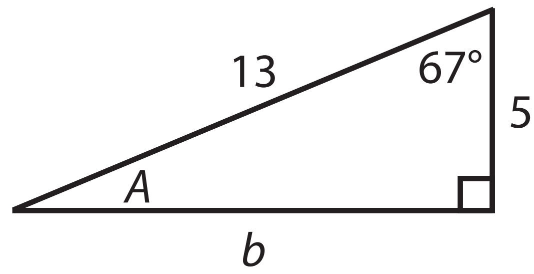
Inverse trigonometric functions find the angle needed to make a given ratio of sides.

Trigonometric functions are used to find lengths of sides, and inverse trigonometric functions are used to find measures of angles.

Inverse Function	Notation	Example *A*	Example *B*
sine inverse	$\sin^{-1}$ or arcsin	$A = \sin^{-1} \frac{5}{13}$	$B = \sin^{-1} \frac{12}{13}$
cosine inverse	$\cos^{-1}$ or arccos	$A = \cos^{-1} \frac{12}{13}$	$B = \cos^{-1} \frac{5}{13}$
tangent inverse	$\tan^{-1}$ or arctan	$A = \tan^{-1} \frac{5}{12}$	$B = \tan^{-1} \frac{12}{5}$

Solving a Right Triangle

Trig equations involve three values: two side lengths and an angle measure. If any two of these values are known, they can be plugged in to solve for the third.



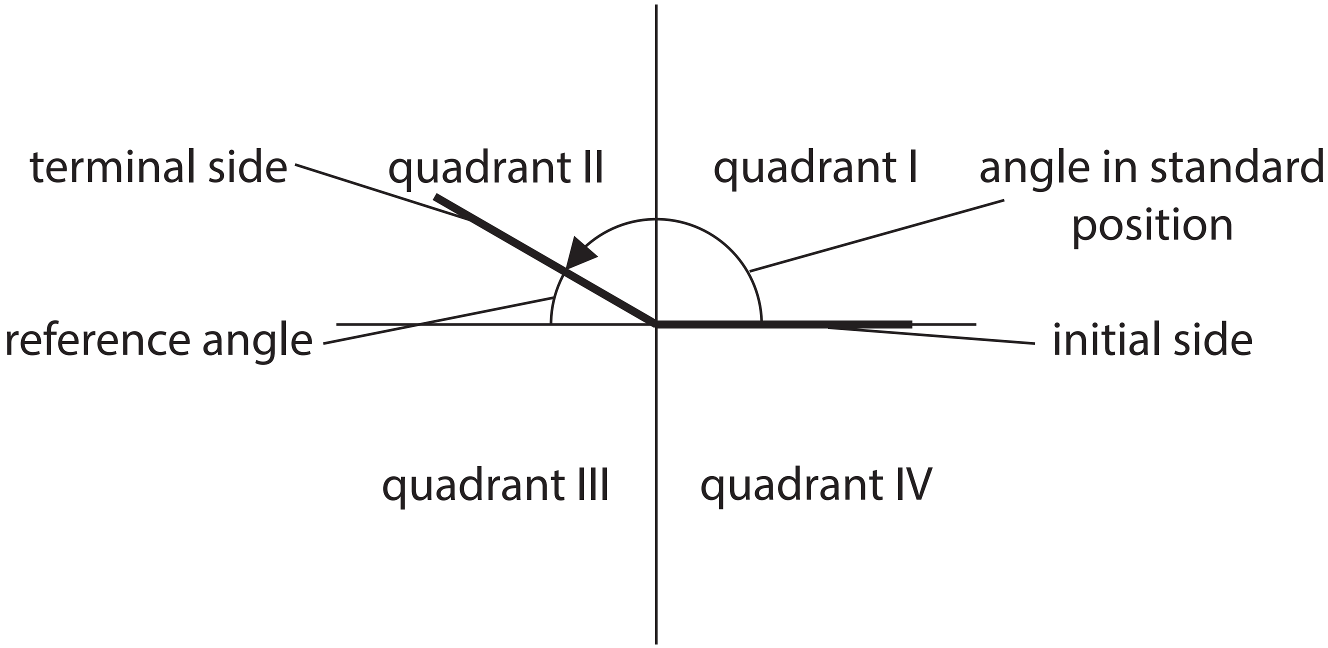
In many cases, more than one trig function can be used. For example, b could also be found by using tangent: $\tan 67° = \frac{b}{5}$.

Part solving for	Method	Sine example
side	Multiply each side of the trig equation by the denominator, and divide by the trig function if necessary. Then evaluate on a calculator.	$\sin 67° = \frac{b}{13}$ $13 \sin 67° = b$ $\mathbf{12 \approx b}$
angle	Do the applicable inverse trig function on each side, to make one side just the angle. Evaluate its value on a calculator.	$\sin A = \frac{5}{13}$ $\sin^{-1} \sin A = \sin^{-1} \frac{5}{13}$ $\mathbf{A \approx 23°}$

Angle Terminology

The terms below are helpful in describing and sketching angles.

Term	Definition
Initial Side	the side of an angle from which it initiates (opens)
Terminal Side	the side of an angle at which it terminates (closes)
Standard Position	position of an angle drawn with its vertex at the origin and its initial side on the positive *x*-axis
Quadrant	quarter of the plane, as divided by the axes
Reference Angle	smallest positive angle between an angle's terminal side and the line the initial side is on

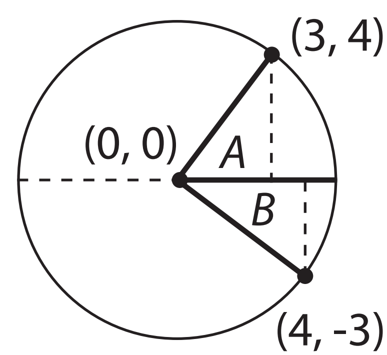


Trigonometric Functions in Circles

The trig function definitions earlier were for angles in right triangles, and thus only apply to angles between 0° and 90°. An expanded definition, that applies to all angles, is based on placing the angle in standard position within a circle centered at (0, 0). The *x* and *y* coordinates of the point where the terminal side intersects the circle are used instead of adjacent and opposite, respectively, and the radius of the circle (which can be found by the Pythagorean theorem) is used instead of hypotenuse.

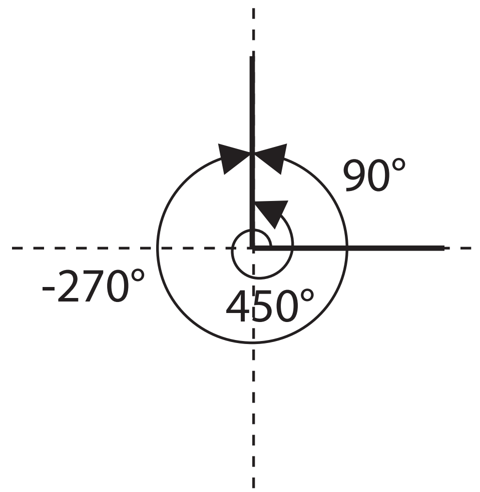
Drawing a vertical line from the point to the *x*-axis makes a right triangle. If the angle is between 0° and 90°, then the definitions are exactly same; otherwise it is the same except that *x*, *y*, or both, are negative.

Function	Ratio	Example *A*	Example *B*
sine	$\sin\theta = \frac{y}{r}$	$\sin A = \frac{4}{5}$	$\sin B = \frac{-3}{5}$
cosine	$\cos\theta = \frac{x}{r}$	$\cos A = \frac{3}{5}$	$\cos B = \frac{4}{5}$
tangent	$\tan\theta = \frac{y}{x}$	$\tan A = \frac{4}{3}$	$\tan B = \frac{-3}{4}$
cotangent	$\cot\theta = \frac{x}{y}$	$\cot A = \frac{3}{4}$	$\cot B = \frac{4}{-3}$
secant	$\sec\theta = \frac{r}{x}$	$\sec A = \frac{5}{3}$	$\sec B = \frac{5}{4}$
cosecant	$\csc\theta = \frac{r}{y}$	$\csc A = \frac{5}{4}$	$\csc B = \frac{5}{-3}$



Coterminal Angles

The measure of an angle can be any size, including negative. An angle not between 0° and 360° has the same terminal side as an angle that is between 0° and 360°, and looks the same. Such identical-looking angles are called **coterminal**, and can make trigonometric problems much simpler. They can be found by adding or subtracting 360° any number of times.

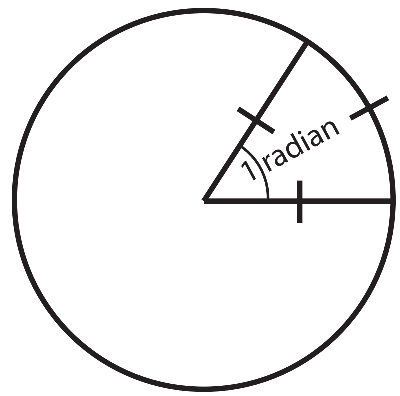


In the example at right, the 90° angle (going up from the *x*-axis), the -270° angle (going down from the *x*-axis), and the 450° angle (going up from the *x*-axis a full circle and then another 90°) are all coterminal.

Angle example	Next three bigger coterminal angles	Next three smaller coterminal angles	Smallest positive coterminal angle
90°	450°, 810°, 1170°	-270°, -630°, -990°	90°
1470°	1830°, 2190°, 2550°	1110°, 750°, 390°	30°

Radians

A **radian** is the measure of an angle in a circle such that the radius of the circle and the arc of the circle are the same length.

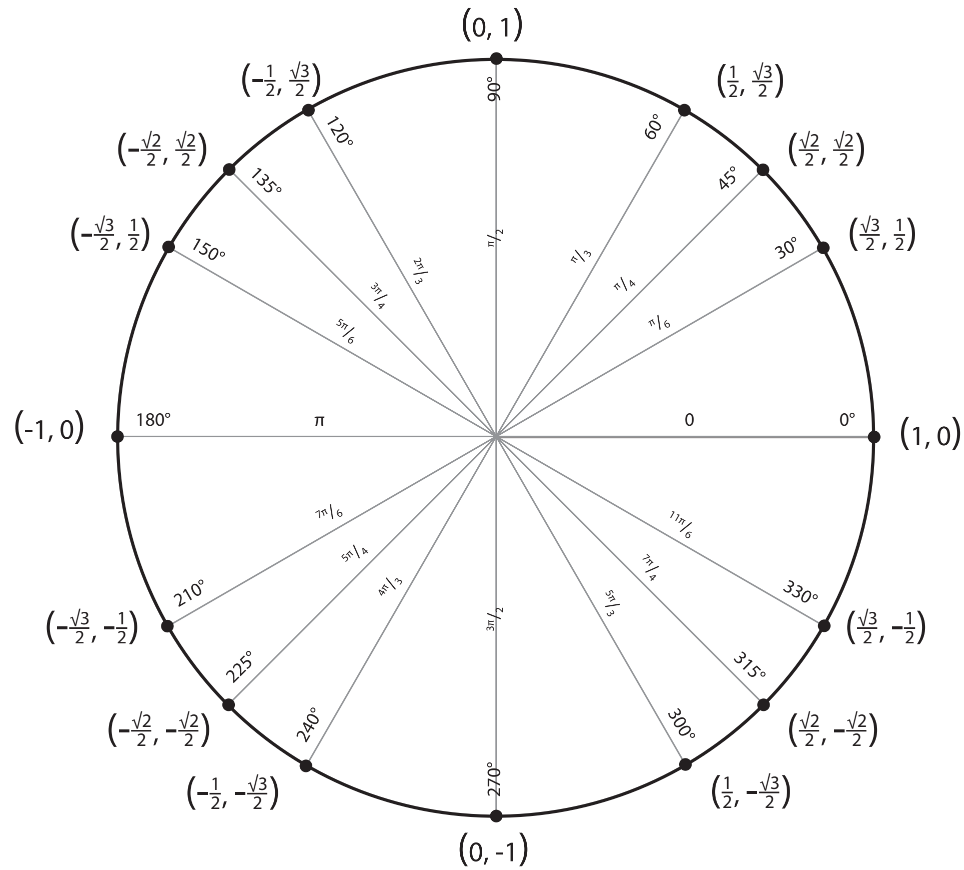


Since the arc length of a full circle (360°) is its circumference ($C = 2\pi r$), **$360° = 2\pi$ radians**. Therefore, radians can be converted to degrees by multiplying by $\frac{360°}{2\pi}$. A simpler method is to apply the conversions in the table below. For example, $\frac{5\pi}{12} = 5(\frac{\pi}{12}) = 5(15°) = 75°$.

Measure in radians	Measure in degrees	Example (x5)	Example (x11)
$\frac{\pi}{12}$	15°	$\frac{5\pi}{12} = 75°$	$\frac{11\pi}{12} = 165°$
$\frac{\pi}{6}$	30°	$\frac{5\pi}{6} = 150°$	$\frac{11\pi}{6} = 330°$
$\frac{\pi}{4}$	45°	$\frac{5\pi}{4} = 225°$	$\frac{11\pi}{4} = 495°$
$\frac{\pi}{3}$	60°	$\frac{5\pi}{3} = 300°$	$\frac{11\pi}{3} = 660°$
$\frac{\pi}{2}$	90°	$\frac{5\pi}{2} = 450°$	$\frac{11\pi}{2} = 990°$
π	180°	$5\pi = 900°$	$11\pi = 1980°$

The Unit Circle

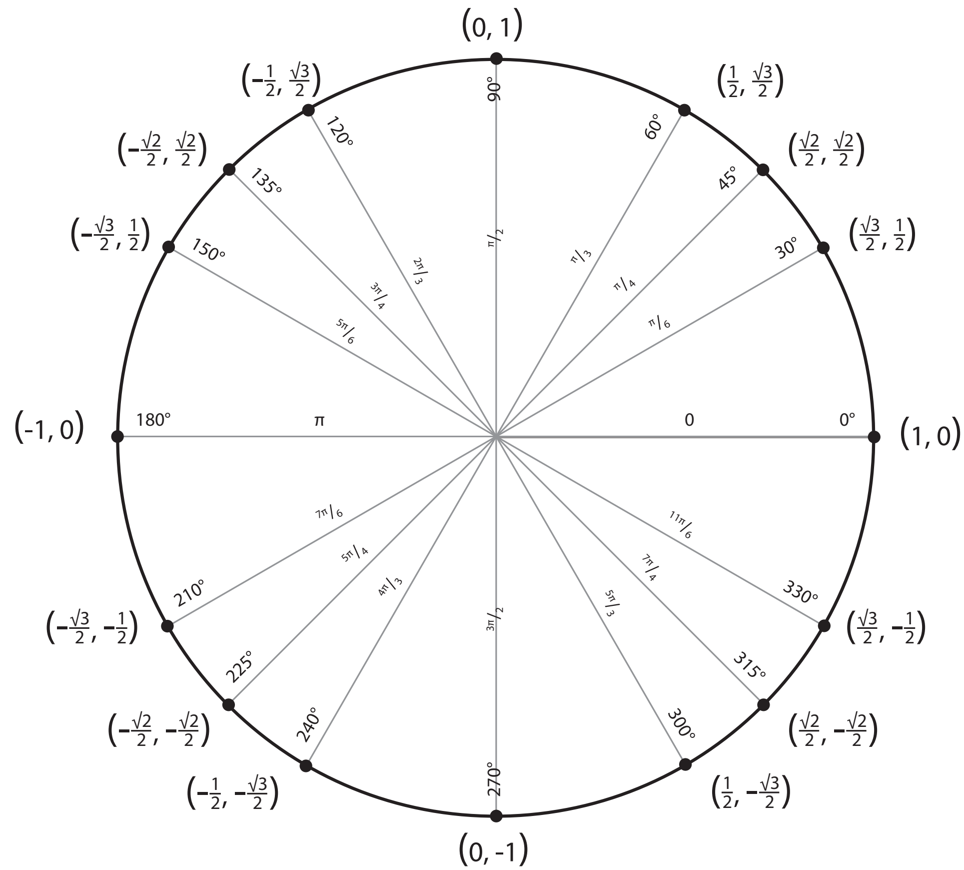
A unit circle is a circle with a radius of 1. They are usually marked in increments of $\frac{\pi}{6}$ (30°) and $\frac{\pi}{4}$ (45°), and can be used to find trig values for these angles, based on the definitions of trig functions in a circle.



Function	Definition	In unit circle	Example: 120°
Sine	$\sin\theta = \frac{y}{r}$	the *y* coordinate	$\frac{\sqrt{3}}{2}$
Cosine	$\cos\theta = \frac{x}{r}$	the *x* coordinate	$\frac{-1}{2}$
Tangent	$\tan\theta = \frac{y}{x}$	$\frac{y}{x}$ (denominators cancel)	$\frac{\sqrt{3}}{2} \div \frac{1}{2} = \frac{\sqrt{3}}{1} = \sqrt{3}$

Inverse Functions and the Unit Circle

Values of inverse trig functions can be found on the unit circle. However, since they are functions, they can only have one output, so their outputs are limited to a 180° range, as shown below. For example, even though 30° and 150° both have a sine of ½, $\sin^{-1} \frac{1}{2} = 30°$ only.



Function	Angle	Range	Example
$\sin^{-1} b$	where $y = b$	$-90° \le A \le 90°$	$\sin^{-1} \frac{-\sqrt{3}}{2} = -60°$
$\cos^{-1} b$	where $x = b$	$0° \le A \le 180°$	$\cos^{-1} \frac{-\sqrt{3}}{2} = 150°$
$\tan^{-1} b$	where $\frac{y}{x} = b$	$-90° \le A \le 90°$	$\tan^{-1} \frac{-\sqrt{3}}{3} = -30°$

Trigonometric Identities

An **identity** is an equation that is always true because the two sides are algebraically equivalent, such as $a + b = a - (\text{-b})$. Common basic trig identities are shown below.

Reciprocal Identity	Proof
$\sin\theta = \frac{1}{\csc\theta}$	$\frac{y}{r} = 1 \div \frac{r}{y}$
$\cos\theta = \frac{1}{\sec\theta}$	$\frac{x}{r} = 1 \div \frac{r}{x}$
$\tan\theta = \frac{1}{\cot\theta}$	$\frac{y}{x} = 1 \div \frac{x}{y}$
$\cot\theta = \frac{1}{\tan\theta}$	$\frac{x}{y} = 1 \div \frac{y}{x}$
$\sec\theta = \frac{1}{\cos\theta}$	$\frac{r}{x} = 1 \div \frac{x}{r}$
$\csc\theta = \frac{1}{\sin\theta}$	$\frac{r}{y} = 1 \div \frac{y}{r}$

Quotient Identity	Proof
$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\frac{y}{x} = \frac{y \div r}{x \div r}$
$\cot\theta = \frac{\cos\theta}{\sin\theta}$	$\frac{x}{y} = \frac{x \div r}{y \div r}$

Pythagorean Identity	Proof
$\sin^2\theta + \cos^2\theta = 1$	$(\frac{y}{r})^2 + (\frac{x}{r})^2 = \frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2}$
$\sec^2\theta - \tan^2\theta = 1$	$(\frac{r}{x})^2 - (\frac{y}{x})^2 = \frac{r^2 - y^2}{x^2} = \frac{x^2}{x^2}$
$\csc^2\theta - \cot^2\theta = 1$	$(\frac{r}{y})^2 - (\frac{x}{y})^2 = \frac{r^2 - x^2}{y^2} = \frac{y^2}{y^2}$
$\sin^2\theta = 1 - \cos^2\theta$	see above
$\cos^2\theta = 1 - \sin^2\theta$	see above
$\tan^2\theta = \sec^2\theta - 1$	see above
$\cot^2\theta = \csc^2\theta - 1$	see above
$\sec^2\theta = 1 + \tan^2\theta$	see above
$\csc^2\theta = 1 + \cot^2\theta$	see above

Double Angle Identities		
$\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = \cos^2\theta - \sin^2\theta$	$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

Verifying Trigonometric Identities

Verifying an identity means establishing that an equation is in fact an identity. This is done by rewriting one side of the equation until it is written the same as the other side. In the example below, the identity $\cot x + \tan x = \csc x \sec x$ is verified.

$\cot x + \tan x$	Justification
$= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$	Rewrite using quotient identities.
$= \frac{\cos x}{\sin x} \bullet \frac{\cos x}{\cos x} + \frac{\sin x}{\cos x} \bullet \frac{\sin x}{\sin x}$	Multiply each term by 1 to get a common denominator.
$= \frac{\cos^2 x}{\sin x \cos x} + \frac{\sin^2 x}{\sin x \cos x}$	Simplify.
$= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x}$	Add.
$= \frac{1}{\sin x \cos x}$	Rewrite the numerator using a Pythagorean identity.
$= \frac{1}{\sin x} \bullet \frac{1}{\cos x}$	Rewrite as separate factors.
$= \csc x \sec x$	Rewrite using reciprocal identities.

Techniques for Verifying Trigonometric Identities

Identities can be verified using basic arithmetic and algebra along with the identities given earlier. There is usually not one specific way to verify a given identity. Commonly, an approach will be tried and then discarded in favor of another.

Several common techniques are listed below. See PreCalculus Chart 2 for details and examples.

Technique	Summary
Rewrite a trigonometric expression without fractions.	Use a reciprocal identity or a quotient identity.
Rewrite a trigonometric expression using only sine and cosine.	Use a reciprocal identity or a quotient identity.
Rewrite a trigonometric expression using a Pythagorean identity.	Use a Pythagorean identity.
Split a fraction with multiple terms in the numerator into separate fractions.	Make a separate fraction for each term in the numerator, using the same denominator.
Add or subtract terms when both are fractions.	Use a common denominator.
Use a conjugate to simplify a trigonometric fraction.	Multiply by a conjugate to get a Pythagorean identity.
Factor a trigonometric expression.	Factor into factors that can be rewritten using identities.