CHAPTER TWO: TRIGONOMETRIC FUNCTIONS — Review October 13 ⚜ Test October 25

In Geometry you learned that the trigonometric functions sine, cosine, and tangent are the ratio of one side of a right triangle to another, such as sine = $^{opposite}/_{hypotenuse}$. This chapter expands on that concept in three ways. First, the reciprocal trig functions cosecant, secant, and cotangent, are the reciprocals of sine, cosine, and tangent, respectively. For example, given sin 59° is $^6/_7$, csc 59° is $^7/_6$. Second, the inverse trig functions $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ are the inverses of sine, cosine, and tangent, respectively. For example, given sin 59° is $^6/_7$, $\sin^{-1}$ $^6/_7$ is 59°. Third, the definitions of the trig functions are expanded to work for all angles, not just those between 0° and 90°. For example, the sine of 59°, 121°, 419°, -239°, and infinitely many other angles is all $^6/_7$. This chapter introduces the concept of radians, which in calculus are usually used instead of degrees to measure angles. Finally, the concepts involved in trigonometric functions are tied together in the form of trigonometric identites, which are trigonometric equations that can be proven to always be true.

2-A Angles in Right Triangles — Thursday • September 22

trigonometric function • sine • cosine • tangent • cotangent • secant • cosecant • inverse trigonometric function • $\sin^{-1}$ (arcsin) • $\cos^{-1}$ (arccos) • $\tan^{-1}$ (arctan)

❶ Find the values of each of the six trigonometric functions of an angle in a right triangle with two known sides.
❷ Find values of cotangent, secant, and cosecant on the calculator.
❸ Calculate a side length in a right triangle based on a known angle and known side length.
❹ Calculate an angle measure in a right triangle based on two known side lengths.
❺ Solve a right triangle.

2-B Angles in Circles — Tuesday • September 27

standard position • coterminal • initial side • terminal side • quadrant • reference angle

❶ Find the values of the trigonometric functions for an angle in standard position that passes through a specific point (x, y).
❷ Find angles coterminal to a given angle.
❸ Sketch an angle in standard position.
❹ Find an angle's reference angle.
❺ Use a reference angle to find the values of the trigonometric functions for an angle θ that is a multiple of 30° or 45°.

2-C Radians — Monday • October 3

radian

❶ Convert between degree and radian measure.
❷ For angles measured in radians, draw sketches, find coterminal angles and reference angles, and find trig ratios.
❸ Find the area of a sector.

2-D The Unit Circle — Thursday • October 6

unit circle

❶ Use a unit circle to find the values of the trigonometric functions for an angle θ that is a multiple of $^{\pi}/_6$ (30°) or $^{\pi}/_4$ (45°).
❷ Use a unit circle to find the value of an inverse trigonometric function for an angle shown in the unit circle.

2-E Trigonometric Identities — Tuesday • October 11

reciprocal identity • quotient identity • Pythagorean identity • double angle identity • conjugate • verify

❶ Rewrite a trigonometric expression without fractions.
❷ Rewrite a trigonometric expression using only sine and cosine.
❸ Rewrite a trigonometric expression using a Pythagorean identity.
❹ Split a fraction with multiple terms in the numerator into separate fractions.
❺ Add or subtract terms when one or both are fractions.
❻ Use a conjugate to simplify a trigonometric fraction.
❼ Factor a trigonometric expression.
❽ Verify a trigonometric identity.

2-A Angles in Right Triangles

The three primary TRIGONOMETRIC Functions are SINE (sin), COSINE (cos), and TANGENT (tan). They show the ratio of the length of one side of a right triangle to the length of another side.

$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

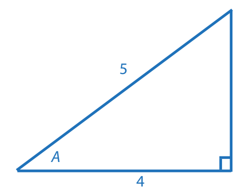
The three reciprocal trigonometric functions are COSECANT (csc), SECANT (sec), and COTANGENT (cot). They are the reciprocals of sine, cosine, and tangent, respectively.

$\frac{1}{\sin\theta} = \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$ $\frac{1}{\cos\theta} = \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$ $\frac{1}{\tan\theta} = \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$

❶ Find the values of each of the six trigonometric functions of a nonright angle in a right triangle with two known sides.

1. Use the Pythagorean theorem $a^2 + b^2 = c^2$ to find the length of the third side.
2. Identify which length is opposite the given angle, which is adjacent to it, and which is the hypotenuse.
3. For each trig function, use the two appropriate lengths in a fraction based on the equations above.

❶ Find the sine, cosine, tangent, cosecant, secant, and cotangent of angle A shown at right.



1. $a^2 + 4^2 = 5^2$
   $a = \sqrt{9} = 3$
2. opposite = 3, adjacent = 4, hypotenuse = 5
3. $\sin A = {}^3/_5$ $\cos A = {}^4/_5$ $\tan A = {}^3/_4$
   $\csc A = {}^5/_3$ $\sec A = {}^5/_4$ $\cot A = {}^4/_3$

Most calculators do not have buttons for the reciprocal trigonometric functions.

❷ Find values of cotangent, secant, and cosecant on the calculator.

1. Type `1/` and then the reciprocal function.

❷ Evaluate sec 25°.

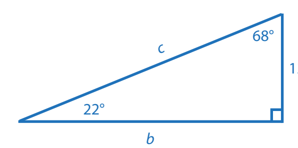
1. $\sec 25° =$ `1/cos(25)` ≈ 1.10

A trigonometric function can be used to calculate an unknown side length from a known angle and known side length in a right triangle.

❸ Calculate a side length in a right triangle based on a known angle and known side length.

1. Write a sine, cosine, or tangent ratio involving the known side, the unknown side, and either nonright angle.
2. Solve for the unknown side.

❸ Solve for c in the triangle at right.



1. $\sin 22° = \frac{12}{c}$
2. $c \sin 22° = 12$
3. $c = \frac{12}{\sin 22°} \approx 32.0$

Note that we could have also started with $\cos 68° = \frac{12}{c}$, but $\sin 68° = \frac{b}{c}$ or $\cos 22° = \frac{b}{c}$ would not have helped because these equations have two unknowns.

Whereas sin, cos, and tan take an angle and find a ratio for it, the INVERSE Trigonometric Functions SIN^{-1}, COS^{-1}, and TAN^{-1} take a ratio and find an angle for it. For example, $\sin 30° = ½$ and $\sin^{-1} ½ = 30°$.

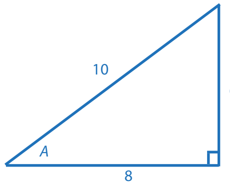
For angles between 0° and 90°, $\sin^{-1} \sin A = A$, $\cos^{-1} \cos A = A$, and $\tan^{-1} \tan A = A$.

$\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ are also called ARCSIN, ARCCOS, and ARCTAN, respectively.

❹ Calculate an angle measure in a right triangle based on two known side lengths.

1. Write a sine, cosine, or tangent ratio involving the angle and the two known sides.
2. Solve for the angle by applying $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ to each side of the equation.

❹ Solve for A in the triangle at right.



1. $\sin A = \frac{6}{10}$
2. $\sin^{-1} \sin A = \sin^{-1} \frac{6}{10}$
3. $A \approx 36.9°$

Note that we could have also started with $\cos A = \frac{8}{10}$ or $\tan A = \frac{6}{8}$.

To solve a triangle is to find every unknown side and angle.

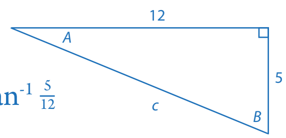
Angles are labeled with capital letters. Each side is labeled with the same letter as the angle opposite it, but lowercase.

❺ Solve a right triangle.

1. If both angles are unknown, solve for one of them using $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ (see ❹).
2. Find the second angle by subtracting the first from 90°.
3. If two sides are unknown, solve for one of them using sin, cos, or tan (see ❸).
4. Find the last side by using sin, cos, or tan, or by using the Pythagorean theorem.

❺ Solve the triangles shown below.

a)



1. $\tan A = \frac{5}{12}$

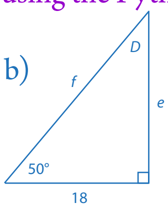
   $\tan^{-1} \tan A = \tan^{-1} \frac{5}{12}$

   $A \approx 22.6°$
2. $B \approx 90° - 22.6° = 67.4°$
3.
4. $\sin 22.6° =$

   $c \sin 22.6° = 5$

   $c = \frac{5}{\sin 22.6°} = 13.0$

b)



$D = 90° - 50° = 40°$

$\tan 50° = \frac{e}{18}$

$18 \tan 50° = e \approx 21.5$

$\cos 50° = \frac{e}{18}$

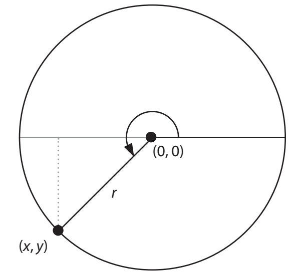
$f \cos 50° = 18$

$f = \frac{18}{\cos 50°} \approx 28.0$

2-B Angles in Circles

An angle is in STANDARD POSITION if its initial side starts at the origin and is on the positive x-axis.

The equations in 2-A for the trigonometric functions only apply to angles between 0° and 90°. They can instead be defined for angles of any measure by placing the angle in standard position in a circle centered at the origin and using the values x, y, and r instead of adjacent, opposite, and hypotenuse. The point (x, y) is where the terminal side intersects the circle, and $r = \sqrt{x^2 + y^2}$ is the radius of the circle.



A vertical line can be drawn from the point (x, y) to the x-axis to create a right triangle with a hypotenuse of r. Unlike in 2-A, x and y can be negative, because they are positions rather than lengths.

$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$

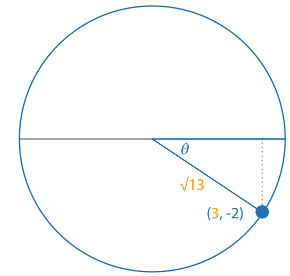
$\csc\theta = \frac{r}{y}$ $\sec\theta = \frac{r}{x}$ $\cot\theta = \frac{x}{y}$

❶ Find the values of the trigonometric functions for an angle in standard position that passes through a specific point (x, y).

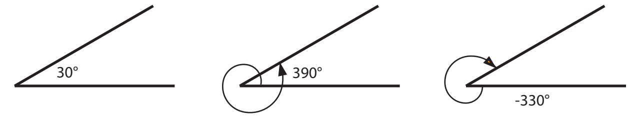
1. Use the Pythagorean theorem to calculate $r = \sqrt{x^2 + y^2}$.
2. Fill in x, y, and r in the ratios above.

❶ Find the cosine of an angle θ in standard position that passes through the point (3, -2).

1. $r = \sqrt{3^2 + (-2)^2} = \sqrt{13}$
2. $\cos\theta = \frac{x}{r} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{3}$



COTERMINAL Angles look the same as each other but are different by one or more full rotations. For example, 390° and -330° are both coterminal to 30° (and to each other).



❷ Find angles coterminal to a given angle.

1. Add or subtract multiples of 360°.

❷ Find three angles coterminal to 240°.

$240° - 360° = -120°$

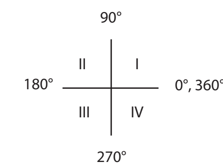
$240° + 360° = 600°$

$240° + 2(360)° = 960°$

Angles are measured counterclockwise from their INITIAL Side to their TERMINAL Side. Angles that open clockwise are negative.

The plane is divided into four QUADRANTs which are labeled I, II, III, and IV in order from 0° to 360°. The axes are not in a quadrant.

❸ Sketch an angle in standard position.



1. Draw the initial side starting at the origin and pointing right.
2. If the angle is not between 0° and 360°, find a coterminal angle that is (see ❷).
3. Use the diagram at right to determine which quadrant the coterminal angle is in or which axis it is on.
4. Determine approximately (or exactly) how much closer the terminal side is to one axis than the other.
5. Draw the terminal side starting at the origin.
6. Draw a curve from the initial side to the terminal side, starting upward for a positive angle or downward for a negative angle.
7. If the original angle is not between -360° and 360°, then divide it by 360°, ignore negatives, and round down. Continue the curve with this many complete rotations in a spiral.
8. Put an arrowhead at the end of the curve, pointing to the terminal side.

❸ Sketch 840° in standard position.

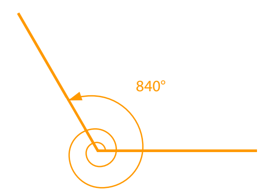
2. $840° - 360° = 480°$

   $480° - 360° = 120°$

3. 120° is in quadrant II.

4. 120° is closer to 90° than to 180°. (To be exact, 120° is ⅓ of the way from 90° to 180°.)

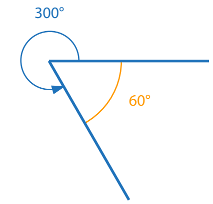
7. $840° \div 360° \approx 2.3$, which rounds down to 2.



The REFERENCE Angle of an angle in standard position is the smallest positive angle between the angle's terminal side and the x-axis.

❹ Find an angle's reference angle.

1. Find the multiple of 180° that is closest to the measure of the angle. (If you are not sure, divide the angle by 180°, round to the nearest integer, and then multiply back by 180°.)
2. Find the positive difference between the original angle and the angle in step 1.



❹ Find the reference angle for a 300° angle.

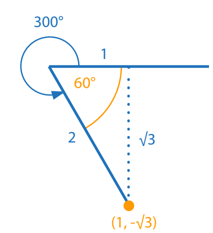
1. 360°
2. 360° – 300° = 60°

❺ Use a reference angle to find the values of the trigonometric functions for an angle θ that is a multiple of 30° or 45°.

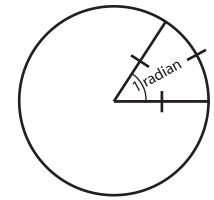
1. Sketch the angle and its reference angle in standard position.
2. If the reference angle is 0, then $\sin\theta = 0$, $\cos\theta = 1$ or -1, and $\tan\theta = 0$.
3. If the reference angle is 90°, then $\sin\theta = 1$ or -1, $\cos\theta = 0$, and $\tan\theta = \frac{1}{0}$ or $\frac{-1}{0}$, which are undefined.
4. Otherwise, draw a vertical line from the terminal side to the x-axis to make a 30°-60°-90° or 45°-45°-90° triangle.
5. Label the hypotenuse with a length of $r = 2$.
6. For a 45°-45°-90° triangle, label side x and side y with a length of $\sqrt{2}$.
   For a 30°-60°-90° triangle, label the short side with a length of 1 and the long side with a length of $\sqrt{3}$, and identify which side is x and which is y.
7. Label the coordinates of the vertex that is not on the axis.
   If the angle is in quadrant II or III, x is negative.
   If the angle is in quadrant III or IV, y is negative.
8. Using $r = 2$ and the values of x and y from the labeled coordinate, fill in x, y, and r in the trigonometric equations and simplify.

❺ Find all six trigonometric ratios for 300°

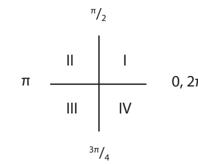
a) $\sin 300° = \frac{\sqrt{3}}{3}$
b) $\cos 300° = \frac{1}{2}$
c) $\tan 300° = \frac{-\sqrt{3}}{1} = -\sqrt{3}$
d) $\csc 300° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$
e) $\sec 300° = \frac{2}{1} = 2$
f) $\cot 300° = \frac{1}{-\sqrt{3}} = \frac{-\sqrt{3}}{3}$



2-C Radians



A RADIAN is the measure of a central angle that intercepts an arc equal in length to the circle's radius.
Since a full circle is 2π radii and 360°, 2π radians = 360° and one radian = $\frac{360°}{2\pi} = \frac{180°}{\pi} \approx 57°$.
An angle measure written without a degree symbol is in radians.
Either radians or degrees can be selected on the calculator after pushing [MODE].



❶ Convert between degree and radian measure.

1. To convert radians to degrees, multiply by $\frac{180°}{\pi}$. It is usually easier to cross-cancel before multiplying.
   To convert degrees to radians, multiply by $\frac{\pi}{180°}$ and write it as a reduced fraction with π in the numerator.

❶ a) Convert $\frac{11\pi}{6}$ to degrees.
$\frac{11\pi}{6} \bullet \frac{180°}{\pi} = \frac{11}{1} \bullet \frac{30°}{1} = 330°$

b) Convert 135° to radians.
$135° \bullet \frac{\pi}{180°} = \frac{135\pi}{180°} = \frac{3\pi}{4}$

If radians are confusing, you can always convert them to degrees. However, most college math uses radians instead of degrees, so it is good to practice doing problems in radians without converting them.

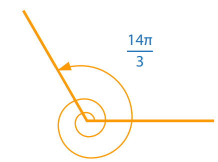
❷ For angles measured in radians, draw sketches, find coterminal angles and reference angles, and find trig ratios.

1. If needed, make sure all angle measures have a common denominator.
2. Follow the appropriate procedure in 2-B. Refer to the quadrant map above or the unit circle in 2-D to visualize angle sizes in radians.

❷ Do the following for the angle $\theta = \frac{14\pi}{3}$.

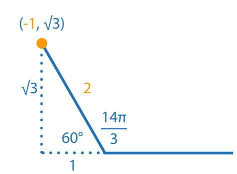
a) Find a coterminal angle (see 2-B ❷).
$\frac{14\pi}{3} + \frac{6\pi}{3} = \frac{20\pi}{3}$

b) Sketch it (see 2-B ❸).
2. $\frac{14\pi}{3} - \frac{6\pi}{3} = \frac{8\pi}{3}$
$\frac{8\pi}{3} - \frac{6\pi}{3} = \frac{2\pi}{3}$



3. $\frac{2\pi}{3}$ is between $\frac{\pi}{2}$ and π, so it is in quadrant II.
4. $\frac{2\pi}{3}$ is closer to $\frac{\pi}{2}$ than to π. (To be exact, $\frac{2\pi}{3}$ is ⅓ of the way from $\frac{\pi}{2}$ to π.)
7. $\frac{14\pi}{3} \div 2\pi = \frac{7}{3}$, which rounds down to 2.

c) Find its reference angle (see 2-B ❹).
$\frac{15\pi}{3}$ is the closest multiple of π
$\frac{15\pi}{3} - \frac{14\pi}{3} = \frac{\pi}{3}$

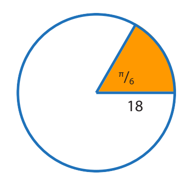


d) Use the reference angle to find the cosine (see 2-B ❺).
6. The reference angle of $\frac{\pi}{3}$ (60°) makes a 30°-60°-90° triangle with the x-axis with sides $x = 1$, $y = \sqrt{3}$, and $r = 2$.
7. $\frac{14\pi}{3}$ is in quadrant II, so the point has a negative x-coordinate and a positive y-coordinate: $(-1, \sqrt{3})$.
8. cosine is $\frac{x}{r}$, so $\cos\frac{14\pi}{3} = \frac{-1}{2}$

The area of a sector of a circle with angle θ and radius r is $A = \frac{1}{2}\theta r^2$. This is because in a full circle $\theta = 2\pi$, making $\pi = \frac{1}{2}\theta$.

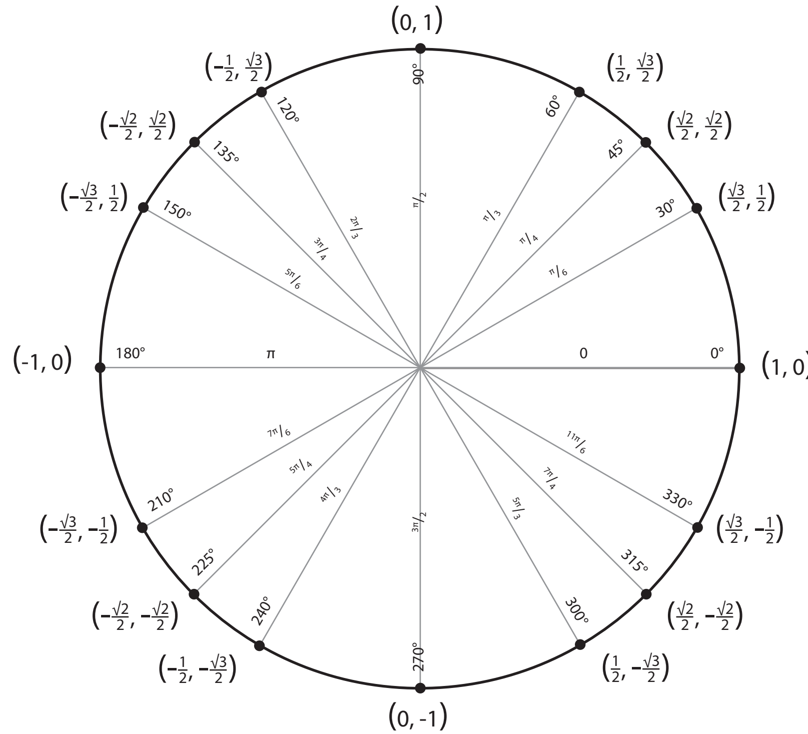
❸ Find the area of a sector.

1. Calculate $\frac{1}{2}\theta r^2$.



❸ Find the area of the sector at right.
1. $A = \frac{1}{2}\left(\frac{\pi}{6}\right)18^2 = 27\pi$

2-D The Unit Circle



A UNIT CIRCLE has a radius of 1. A labeled unit circle can be used to find the values of trig functions without a calculator.

❶ Use the unit circle to find the values of the trigonometric functions for an angle θ that is a multiple of $\frac{\pi}{6}$ (30°) or $\frac{\pi}{4}$ (45°).

1. If the angle is not between 0 and 2π (0° and 360°), change it to a coterminal angle that is.
2. Find the angle in the unit circle and note its (x, y) coordinates.
3. Fill in x, y, and $r = 1$ in the appropriate ratio (see 2-B ❶), and simplify. For tan and cot, the 2's in the denominators will cancel.

❶ Find a) cos 120°, b) tan 150°, c) sec $\frac{3\pi}{2}$, and d) sin $\frac{10\pi}{3}$.

a) $\cos 120° = \frac{x}{r} = \frac{-1/2}{1} = -1/2$

b) $\tan 150° = \frac{y}{x} = \frac{1/2}{-\sqrt{3}/2} = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

c) $\sec \frac{3\pi}{2} = \frac{r}{x} = \frac{1}{0}$, which is undefined.

d) $\frac{10\pi}{3} > 2\pi$

$\frac{10\pi}{3} - \frac{6\pi}{3} = \frac{4\pi}{3}$

$\sin \frac{10\pi}{3} = \sin \frac{4\pi}{3} = \frac{y}{r} = \frac{-\sqrt{3}/2}{1} = -\frac{\sqrt{3}}{2}$

The domain of $\sin^{-1}$ and of $\cos^{-1}$ is $-1 \le x \le 1$, because this is the range of sine and of cosine.

Different angles have the same sine as each other. For example, sin 30° = sin 390° = ½.

To make $\sin^{-1}$ a function, it is defined to be only the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (-90° and 90°) with the given sine. Therefore, $\sin^{-1}$ ½ is only 30°, not also 390°, even though sin 390° = ½.

The same is true for $\tan^{-1}$. For $\cos^{-1}$, the range is angles between 0 and π (0° and 180°).

❷ Use the unit circle to find the value of an inverse trigonometric function for an angle shown in the unit circle.

1. For $\sin^{-1} a$, find the angle in quadrant I or IV or on an adjacent axis that has a y value of a in the unit circle.
For $\cos^{-1} a$, find the angle in quadrant I or II or on an adjacent axis that has an x value of a in the unit circle.
For $\tan^{-1} a$, find the angle in quadrant I or IV or on an adjacent axis for which $\frac{y}{x} = a$. (2's in the denominators will cancel.)
2. If the angle is in quadrant IV or on the negative y axis, subtract 2π (360°) from it.

❷ Find the following in degrees.

a) $\sin^{-1} \frac{1}{2}$	b) $\cos^{-1} \frac{1}{2}$	c) $\sin^{-1} -\frac{1}{2}$	d) $\tan^{-1} -1$	e) $\tan^{-1} \frac{\sqrt{3}}{3}$	f) $\cos^{-1} 2$
1. 30°	60°	330°	$-1 = \frac{-\sqrt{2}}{\sqrt{2}}$, so 315°	$\frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$, so 30°	not possible:
2.		330° – 360° = -30°	315° – 360° = -45°		2 is not in the domain

❷ Find the following in radians.

a) $\sin^{-1} \frac{\sqrt{2}}{2}$	b) $\tan^{-1} -1$
1. $\frac{\pi}{4}$	$-1 = \frac{-\sqrt{2}}{\sqrt{2}}$, so $\frac{7\pi}{4}$
2.	$\frac{7\pi}{4} - \frac{8\pi}{4} = \frac{-\pi}{4}$

2-E Trigonometric Identities

A TRIGONOMETRIC IDENTITY is a trigonometric equation that is always true because the two sides are algebraically equivalent. The most common are shown below.

There are three pairs of RECIPROCAL Identities:

$\sin\theta = \frac{1}{\csc\theta}$ $\cos\theta = \frac{1}{\sec\theta}$ $\tan\theta = \frac{1}{\cot\theta}$

$\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

There is one pair of QUOTIENT Identities:

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

There are three sets of PYTHAGOREAN Identities:

$\sin^2\theta + \cos^2\theta = 1$ $\sec^2\theta - \tan^2\theta = 1$ $\csc^2\theta - \cot^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$ $\sec^2\theta = \tan^2\theta + 1$ $\csc^2\theta = \cot^2\theta + 1$

$\cos^2\theta = 1 - \sin^2\theta$ $\tan^2\theta = \sec^2\theta - 1$ $\cot^2\theta = \csc^2\theta - 1$

There are three DOUBLE ANGLE Identities:

$\sin 2\theta = 2\sin\theta\cos\theta$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

Trigonometric functions raised to a power are conventionally written with the power directly on the function, such as $\sin^2 x$ instead of $(\sin x)^2$.

Trigonometric expressions can be simplified and rewritten using the trigonometric identities and algebraic manipulation.

❶ Rewrite a trigonometric expression without fractions.

1. Factor the fraction into separate fractions that have reciprocal and quotient identities.
2. Rewrite these fractions using the appropriate reciprocal and quotient identities.

❶ Simplify $\frac{\sin x}{\cos^2 x}$.

1. $\frac{\sin x}{\cos x} \bullet \frac{1}{\cos x}$
2. $\tan x \sec x$

❷ Rewrite a trigonometric expression using only sine and cosine.

1. Use the reciprocal and quotient identities for $\csc x$, $\sec x$, $\tan x$, and $\cot x$, and simplify.

❷ Rewrite $\csc^2 x \cot x$.

1. $\frac{1\,x}{\sin^2 x} \bullet \frac{\cos x}{\sin x} = \frac{\cos x}{\sin^3 x}$

❸ Rewrite a trigonometric expression using a Pythagorean identity.

1. Use a Pythagorean identity to rewrite a squared trig function.

❸ Rewrite using a Pythagorean identity.

a) $6\cot^2 x + 2$

1. $6(\csc^2 x - 1) + 2$

$6\csc^2 x - 4$

b) $5\cos x - 3\sin^2 x$

$5\cos x - 3(1 - \cos^2 x)$

$3\cos^2 x + 5\cos x + 3$

Fractions can be algebraically split apart or combined together.

❹ Split a fraction with multiple terms in the numerator into separate fractions.

1. Put each term in the numerator in its own fraction over the denominator.

❹ Write $\frac{1-\sin x}{\cos x}$ as two separate fractions.

1. $\frac{1}{\cos x} - \frac{\sin x}{\cos x}$

❺ Add or subtract terms when one or both are fractions.

1. Get a common denominator by multiplying one or more of the terms by a fraction with a numerator and denominator that are the same.
2. Add or subtract the numerators.

❺ Subtract $\frac{1}{\sin x} - \frac{\sin x}{\tan x}$.

1. $\frac{1}{\sin x} \bullet \frac{\tan x}{\tan x} - \frac{\sin x}{\tan x} \bullet \frac{\sin x}{\sin x} = \frac{\tan x}{\sin x \tan x} - \frac{\sin^2 x}{\sin x \tan x}$
2. $\frac{\tan x - \sin^2 x}{\sin x \tan x}$

The CONJUGATE of $a + b$ is $a - b$. A binomial multiplied by its conjugate is $(a+b)(a-b) = a^2 - b^2$.

❻ Use a conjugate to simplify a trigonometric fraction.

1. Multiply the numerator and denominator by the conjugate of the denominator.
2. If possible, use a Pythagorean identity to simplify the denominator.
3. Continue simplifying.

❻ Simplify $\frac{\cos x}{1-\sin x}$.

1. $\frac{\cos x}{1-\sin x} \bullet \frac{1+\sin x}{1+\sin x} = \frac{\cos x + \sin x \cos x}{1-\sin^2 x}$
2. $\frac{\cos x + \sin x \cos x}{\cos^2 x}$
3. $\frac{\cos x}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x} = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \sec x + \tan x$

Trig expressions can be factored like other expressions.

❼ Factor a trigonometric expression.

1. Factor normally, treating the trig function and its argument (e.g., "tan x") as a single variable like x.

❼ Factor.

a) $x^2 + 8x + 15$
$(x+3)(x+5)$

b) $\tan^2 x + 8\tan x + 15$
$(\tan x + 3)(\tan x + 5)$

c) $2x^3 + 16x + 30$
$2x(x+3)(x+5)$

d) $2\tan^3 x + 16\tan^2 x + 30\tan x$
$2\tan x(\tan x + 3)(\tan x + 5)$

VERIFYING a Trigonometric Identity means showing that one side of the equation is equivalent to the other side. This is done by rewriting one or both sides of the equation, such as by using a Pythagorean identity. Do not change the value of either side, such as by dividing each side by sin *x*.

❽ Verify a trigonometric identity.

1. Choose one side of the identity to manipulate. It is usually easier to choose the more complicated side.
2. Apply one or more of the seven methods described above:

❶ Rewrite a trigonometric expression without fractions.
❷ Rewrite a trigonometric expression using only sine and cosine.
❸ Rewrite a trigonometric expression using a Pythagorean identity.
❹ Split a fraction with multiple terms in the numerator into separate fractions.
❺ Add or subtract terms when one or both are fractions.
❻ Use a conjugate to simplify a trigonometric fraction.
❼ Factor a trigonometric expression.

3. If it is helpful, do steps 1 and 2 for the other side of the identity as well.

❽ Verify the following identities, and label each step with one of the methods ❶ through ❼ shown above.

a) $\cot x + \tan x = \csc x \sec x$

❷ $\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$

❺ $\frac{\cos x}{\sin x} \bullet \frac{\cos x}{\cos x} + \frac{\sin x}{\cos x} \bullet \frac{\sin x}{\sin x} = \frac{\cos^2 x}{\sin x \cos x} + \frac{\sin^2 x}{\sin x \cos x} = \frac{\cos^2 x + \sin^2 x}{\sin x \cos x}$

❸ $\frac{1}{\sin x \cos x}$

❶ $\frac{1}{\sin x} \bullet \frac{1}{\cos x} = \csc x \sec x$

b) $\frac{1}{\sin x - \sin x \cos x} = \csc^3 x + \cot x \csc^2 x$

❼ $\frac{1}{\sin x\,(1 - \cos x)}$

❻ $\frac{1}{\sin x\,(1 - \cos x)} \bullet \frac{1 + \cos x}{1 + \cos x} = \frac{1 + \cos x}{\sin x\,(1 - \cos^2 x)} = \frac{1 + \cos x}{\sin x\,(\sin^2 x)} = \frac{1 + \cos x}{\sin^3 x}$

❹ $\frac{1}{\sin^3 x} + \frac{\cos x}{\sin^3 x}$

❶ $\frac{1}{\sin^3 x} + \frac{\cos x}{\sin x} \bullet \frac{1}{\sin^2 x} = \csc^3 x + \cot x \csc^2 x$