CHAPTER SIX: PROBABILITY

Review February 10 ⇝ Test February 28

The probability of a simple event is the number of ways the event can take place divided by the total number of equally likely possible outcomes, based on the information known at the time. For example, the probability of drawing an ace from a deck of cards with two jacks missing is $^4/_{50}$ because there are 50 equally likely cards that can be drawn, and four of them are aces. More complicated problems arise from situations involving multiple events, but even simple probability problems tend to confuse people. One reason is that people have a hard time not making assumptions about unknown information. For example, a common misconception is that the first card in a deck has a $^4/_{52}$ chance of being an ace and the second card in a deck has a $^4/_{51}$ chance of being an ace, when in reality every card in the deck is equally likely to be an ace.

6-A Counting Methods — Tuesday • 1/24

Pascal's triangle • factorial • choose • combination • sample space • fundamental counting principle • permutation

1. Write the first n rows of Pascal's triangle.
2. Write the n^{th} row of Pascal's triangle.
3. Calculate $n!$
4. Find the r^{th} element in the n^{th} row of Pascal's triangle.
5. Count combinations.
6. Find the total number of possible outcomes in a series of events.
7. Count permutations.
8. Find the size of a sample space by using permutations, if possible.

6-B Probability of a Single Event — Friday • 1/27

mutually exclusive

1. Read set notation.
2. Use the size of a sample space to find the probability of an event.
3. Find the probability of either of two events.

6-C Conditional Probability — Tuesday • 1/31

given • conditional probability

1. Find probabilities based on given information.
2. Make a table to calculate conditional probabilities for two events.

6-D Probability of Specific Multiple Events — Friday • 2/3

dependent events • independent events

1. Identify whether events are dependent or independent.
2. Calculate the probability of multiple events.

6-E Probability of General Multiple Events — Wednesday • 2/8

1. Calculate the probability of an event that can occur in different ways.
2. Calculate the probability of an event that can occur in different orders.
3. Calculate the probability of at least or at most x out of n occurrences of an event.

6-A Counting Methods

Rows 0 through 10 of PASCAL'S TRIANGLE are shown at right.

										1										
									1		1									
								1		2		1								
							1		3		3		1							
						1		4		6		4		1						
					1		5		10		10		5		1					
				1		6		15		20		15		6		1				
			1		7		21		35		35		21		7		1			
		1		8		28		56		70		56		28		8		1		
	1		9		36		84		126		126		84		36		9		1	
1		10		45		120		210		252		210		120		45		10		1

➊ Write the first n rows of Pascal's triangle.

1. Write 1's down the outside as shown at right.
2. Fill in each number inside the triangle as the sum of the two numbers above it.

➋ Write the n^{th} row of Pascal's triangle.

1. Element 0 is 1.
2. To find element 1, multiply by n and divide by 1.
3. To find the next element, multiply by 1 less than before and divide by 1 more than before.
4. Repeat step 3 until the last element (element n, which is 1) is found.

❷ Write row 7 of Pascal's triangle.

1. 1
2. $1 \bullet \frac{7}{1} = 7$
3. $7 \bullet \frac{6}{2} = 21$
4. $21 \bullet \frac{5}{3} = 35$

$35 \bullet \frac{4}{4} = 35$

$35 \bullet \frac{3}{5} = 21$

$21 \bullet \frac{2}{6} = 7$

$7 \bullet \frac{1}{7} = 1$

n FACTORIAL is the product $n! = 1 \bullet 2 \bullet 3 \bullet \ldots \bullet n$.

Since $\frac{n!}{n} = (n-1)!$, $0! = \frac{1!}{1} = 1$.

➌ Calculate $n!$

1. $0! = 1$. For $n > 0$, multiply together each integer from 1 to n.

❸ 5!

$1 \bullet 2 \bullet 3 \bullet 4 \bullet 5 = 120$

n CHOOSE r is the r^{th} element in the n^{th} row of Pascal's triangle. It is written $\binom{n}{r}$ or ${}_nC_r$.

The formula for n choose r, based on ❷, is $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$.

➍ Find the r^{th} element in the n^{th} row of Pascal's triangle.

1. Write out the factors of $n!$ in the numerator and write out the factors of $r!$ and of $(n-r)!$ in the denominator.
2. Cancel the factors of $r!$ or of $(n-r)!$ with the same factors in the numerator. (Or do not write these factors in the first place.)
3. Multiply and divide the remaining factors.

❹ Find element 2 in row 7 of Pascal's triangle.

$\binom{7}{2} = \frac{7!}{5! \bullet 2!} = \frac{7 \bullet 6 \bullet \cancel{5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}}{1 \bullet 2 \bullet \cancel{5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}} = \frac{42}{2} = 21$

A COMBINATION is a group of elements not assigned specific labels or placements within the group. The number of possible combinations of r elements chosen from a group of n elements is $\binom{n}{r}$.

❺ Count combinations.

1. Calculate $\binom{n}{r}$ (see ❹).

❺ In how many ways can Naomi choose her 3 favorite songs from a playlist of 9 songs?

$\binom{9}{3} = \frac{9!}{3! \cdot 6!} = \frac{9 \cdot 8 \cdot 7 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{1 \cdot 2 \cdot 3 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = \frac{504}{6} = 84$

A SAMPLE SPACE consists of all the possible outcomes of an event, such as "♠, ♥, ♦, ♣" for suit of a card, or "♠♠♦, ♠♦♠, ♦♠♠" for a combination of two spades and a diamond.

The size of a sample space can be found using choose. In the examples above, there are $\binom{4}{1} = 4$ ways to choose one of the four suits, and there are $\binom{3}{2} = 3$ ways to choose which 2 out of 3 cards are spades.

The FUNDAMENTAL COUNTING PRINCIPLE states that the total number of possible outcomes of a series of events is the product of the sizes of the individual sample spaces.

❻ Find the total number of possible outcomes in a series of events.

1. Identify the size of the sample space of each individual event, using $\binom{n}{r}$ as needed.
2. Multiple these sizes together.

❻ State the number of possible outcomes of the following.

a) Choose 3 representatives out of 9 seniors and 2 representatives out of 8 juniors.

$\binom{9}{3}\binom{8}{2} = 84 \cdot 28 = 2352$

b) Identify the 1^{st} place, 2^{nd} place, 3^{rd} place, and 4^{th} place finisher out of 25 racers.

$\binom{25}{1}\binom{24}{1}\binom{23}{1}\binom{22}{1} = 25 \cdot 24 \cdot 23 \cdot 22 = 303600$

The outcome of a series of events that each involve choosing one element from those remaining, as in example ❻b above, is called a PERMUTATION. This is the same as a combination, except that each item in the group is assigned a specific label or placement within the group. In other words, a combination is a selection of r items from a group of n items, and a permutation is a selection of r items, *chosen one at a time*, from a group of n items.

Permutations are never required, but can be used as an easier way to indicate multiple combinations in which one of the remaining elements is chosen each time. For example, $\binom{9}{1}\binom{8}{1}\binom{7}{1}$ can be written as ${}_9P_3$.

The number of permutations of r elements that can be made from a set of n elements is $nPr = \frac{n!}{(n-r)!}$.

❼ Count permutations.

1. Write out $n!$ in the numerator and write out $(n-r)!$ in the denominator.
2. Cancel the terms of $(n-r)!$ with the same terms in the numerator. (Or do not write these terms in the first place.)
3. Multiply the remaining terms.

❼ In how many ways can Naomi choose her favorite, second favorite, and third favorite song from a playlist of 9 songs?

${}_9P_3 = \frac{9!}{3! \cdot 6!} = \frac{9 \cdot 8 \cdot 7 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = 504$

❽ Find the size of a sample space by using permutations, if possible.

1. Express the number of possible outcomes as a series of combinations.
2. If each combination involves removing a single item, that is, $\binom{n}{1}\binom{n-1}{1}\binom{n-2}{1}\binom{n-3}{1}\ldots$, the expression can be rewritten as nPr, where r is the total number of items being chosen. Otherwise, permutations do not apply.

❽ Use combinations to express the size of the sample space for each of the following. Then rewrite the solution using permutations if possible, or explain why not.

a) Heather chooses her 3 favorite months.

$\binom{12}{3}$ cannot be written as a permutation, because she is choosing a group of three equal items, not three separate, distinguishable items.

b) Heather chooses her favorite, second favorite, and third favorite month.

$\binom{12}{1}\binom{11}{1}\binom{10}{1}$ can be written ${}_{12}P_3$.

c) Use a 6-sided die to choose a color for each of three teams.

$\binom{6}{1}\binom{6}{1}\binom{6}{1}$ cannot be written as a permutation, because the colors are not being removed as they are chosen.

d) Put 6 colors in a hat and draw three of them to choose a different color for each of three teams.

$\binom{6}{1}\binom{5}{1}\binom{4}{1}$ can be written ${}_6P_3$.

6-B Probability of a Single Event

A SET is a collection of items, called ELEMENTS. $x \in A$ means x is an element of set A.

The CARDINALITY of a Set A, $|A|$, is the number of elements it contains.

The INTERSECTION of Two Sets A and B, $A \cap B$, is the set of elements that are in both A and B.

The UNION of Two Sets A and B, $A \cup B$, is the set of elements that are in either A or B (or both).

The COMPLEMENT of a Set A, A', is the set of elements not in A. Other notations, such as $\overline{A}$, are sometimes used as well.

The UNIVERSAL Set, U, is the set of all elements in the sample space.

The EMPTY Set, $\varnothing$, contains no elements.

❶ Read set notation.

1. Use the definitions above. In general, $\cap$ means *and*, $\cup$ means *or*, ’ means *not*, and $|\ \ |$ means *number of*.

❶ State the following in words, given A is the set of aces and B is the set of black cards.

a) A	the aces
b) U	all of the cards
c) $\varnothing$	none of the cards
d) $\lvert A\rvert$	the number of aces
e) $\lvert U\rvert$	the number of cards
f) $A \cup B$	the cards that are an ace or black
g) $A \cap B$	the cards that are an ace and black
h) $\lvert A \cap B\rvert$	the number of black aces
i) A'	the cards that are not aces
j) $(A \cap B)'$	the cards that are not black aces
k) $(A \cup B)'$	the cards that are not an ace or black

In probability, a set is used to represent an event, with the elements in the set being the event’s possible outcomes.

The probability of event A, $P(A)$, is the probability of an outcome in A occuring. $P(A) = \frac{|A|}{|U|}$.

Because $|A|$ and $|U|$ have specific meanings, reducing a probability or converting it to a decimal or a percent causes information to be lost. Therefore, unless directed otherwise, use only fractions for probabilities in this class, and do not reduce probabilities not equal to 0 or 1.

❷ Use the size of a sample space to find the probability of an event.

1. Identify the denominator $|U|$, the size of the sample space. Use combinations if needed (see ❷).
2. Identify the numerator $|A|$, which is the number of possible outcomes for event A within the sample space. Use combinations if needed (see ❷).

❷ Ryan draws two cards. Find the probability that …

a) the first card is a ace

1. There are $|U| = 52$ possible cards.
2. There are $|A| = 4$ possible aces.
3. $P(A) = \frac{4}{52}$

b) both cards are aces

There are $|U| = \binom{52}{2} = 1326$ possible combinations of 2 of the 52 cards.

There are $|A| = \binom{4}{2} = 6$ possible combinations of 2 of the 4 aces.

$P(A) = \frac{6}{1326}$

Two events are said to be MUTUALLY EXCLUSIVE, or Disjoint, if the occurance of one eliminates the possibility of the other, such as clubs and hearts on a single card. If A and B are mutually exclusive, the probability that one of them will happen is $P(A \cup B) = P(A) + P(B)$.

The above formula does not work for events that are not mutually exclusive, because some outcomes would be counted more than once. To account for this, the overlap between the events can be subtracted: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

❸ Find the probability of either of two events.

1. Add the probabilities of the two events.
2. Subtract the probability of the two events happening simultaneously, since this has been double-counted.

❸ Find the probability of a card being as stated.

a) red or an ace

1. $\frac{26}{52} + \frac{4}{52} = \frac{30}{52}$
2. $\frac{30}{52} - \frac{2}{52} = \frac{28}{52}$

b) a 9 or an ace

$\frac{4}{52} + \frac{4}{52} = \frac{8}{52}$

$\frac{8}{52} - 0 = \frac{8}{52}$

6-C Conditional Probability

Probability is based on what is known, not on what has happened. GIVEN means *known* (or *knowing*). Given information changes the universal set. For example, given a card is red, the probability of it being hearts is $\frac{13}{26}$, because U now only includes the 26 red cards.

CONDITIONAL Probabilities incorporate given information that would not necessarily be assumed, such as *the card is red*. The probability of event A after knowledge of event B has been taken into account is the probability of A given B: $P(A \mid B) = \frac{|A \cap B|}{|B|}$.

❹ Find probabilities based on given information.

1. Do not assume any information, even for events that have already occurred.
2. Take into account all known (given) information, even for events that have not yet occurred.

❹ Find the following probabilities for Ryan's two cards.

a) The second card is an ace.

1. The first card remains unknown. Do no assume it is or is not an ace. There are still $|U| = 52$ possible outcomes, $|A| = 4$ of which are aces, so $P(A) = \frac{4}{52}$.

b) The second card is an ace, given the first card is an ace.

2. The first card is known, so use that information. There are only $|U| = 51$ remaining possible outcomes for the second card, $|A| = 3$ of which are aces, so $P(A) = \frac{3}{51}$.

c) The first card is an ace, given the second card will be an ace.

2. The second card is known even though it hasn't been flipped yet, so use that information. There are only $|U| = 51$ remaining possible outcomes for the first card, $|A| = 3$ of which are aces, so $P(A) = \frac{3}{51}$.

❺ Make a table to calculate conditional probabilities for two events.

1. Make a row for A and a row for A'. Label them in context.
2. Make a column for B and a column for B'. Label them in context.
3. Multiply to find $P(A \cap B)$, $P(A' \cap B)$, $P(A \cap B')$, and $P(A' \cap B')$, and put these values in the table.
4. Add to find $P(A)$, $P(A')$, $P(B)$, and $P(B')$, and put these values in the table. Verify that $P(A) + P(A')$ 100% and $P(B) + P(B') = 100\%$.
5. Use the formula for conditional probability.

❺ 10% percent of the population has a certain disease. A test for this disease gives a positive result for 70% of people who have the disease and for 20% of people who do not. Show this information in a table, and use it to calculate the probabilities below.

	B: tests positive	B': tests negative	Total
A: has disease	$P(A \cap B) = 10\% \bullet 70\% = 7\%$	$P(A \cap B') = 10\% \bullet 30\% = 3\%$	$P(A) = 10\%$
A': does not have disease	$P(A' \cap B) = 90\% \bullet 20\% = 18\%$	$P(A' \cap B') = 90\% \bullet 80\% = 72\%$	$P(A') = 90\%$
Total	$P(B) = 25\%$	$P(B') = 75\%$	100%

a) a person tests positive: $P(B) = 25\%$

b) a person has the disease and tests positive: $P(A \cap B) = 7\%$

c) a person has the disease, given he or she tests positive: $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{7}{25} = 28\%$

6-D Probability of Specific Multiple Events

The probabilities of DEPENDENT Events are conditional: They are influenced by each others' outcomes.

The probabilities of INDEPENDENT Events do not change.

❶ Identify whether events are dependent or independent.

1. If the probability of events are the same on every trial, they are independent.
2. If the outcome of an event changes the sample space for other events, they are dependent.

❶ Juwan rolls three dice, checking for a 6 each time, and she draws two cards, checking for an ace each time.

1. The die rolls are independent: Each die has a $\frac{1}{6}$ chance of rolling a 6, regardless of the previous rolls.
2. The card draws are dependent: After she knows the first card, there are only 51 remaining possibilities for the second card. The second card has a $\frac{4}{51}$ chance of being an ace if the first card was not an ace, or a $\frac{3}{51}$ chance of being an ace if the first card was an ace.

The probability of both A and B occuring is the product of their probabilities: $P(A \cap B) = P(A) \bullet P(B)$. If the events are dependent, the probability of one of them must be adjusted to its conditional probability based on the other event: $P(A \cap B) = P(A) \bullet P(B \mid A)$, or equivalently, $P(B) \bullet P(A \mid B)$.

❷ Calculate the probability of multiple events.

1. List what has to occur. Ignore events that do not matter.
2. Identify the conditional probability of each event, given each of the previous events occuring.
3. Multiply all the probabilities together.

❷ Davin draws five cards. Calculate the probability that the first two cards are aces and the fourth card is not an ace.

1. ace, ace, not ace
   (Note that the third and fifth card are irrelevant to this problem.)
2. $P(\text{first card is an ace}) = \frac{4}{52}$
   $P(\text{second card is an ace, given first card is an ace}) = \frac{3}{51}$
   $P(\text{fourth card is not an ace, given first two cards are aces}) = \frac{48}{50}$
3. $\frac{4}{52} \bullet \frac{3}{51} \bullet \frac{48}{50} = \frac{576}{132600}$

6-E Probability of General Multiple Events

Some events can take place in different ways. To calculate the total probability, the probability of each of the possible ways is added together.

❶ Calculate the probability of an event that can occur in different ways.

1. Identify each different way it can occur.
2. Calculate the probability of each way (see 6-D).
3. Add these probabilities together.

❶ Kyley grabs 3 random pens from a drawer with 6 black pens, 4 red pens, and 1 purple pen. What is the probability that they are all the same color?

1. It could be black, black, black, or it could be red, red, red.
2. $P(\text{all black}) = \frac{6}{11} \bullet \frac{5}{10} \bullet \frac{4}{9} = \frac{120}{990}$
   $P(\text{all red}) = \frac{4}{11} \bullet \frac{3}{10} \bullet \frac{2}{9} = \frac{24}{990}$
3. $P(\text{all black or all red}) = \frac{120}{990} + \frac{24}{990} = \frac{144}{990}$

When the different ways in which an event can happen are simply different orders of the same thing, each way will have the same probability. Therefore, instead of separately identifying each order and calculating its probability, the probability can be calculated once and multiplied by the number of possible orders.

❷ Calculate the probability of an event that can occur in different orders.

1. Identify one possible order.
2. Calculate the probability of this order.
3. Count or calculate the number of possible orders, using $\binom{n}{r}$ as needed.
4. Multiply the number of possible orders by the probability of each order.

❷ What is the probability that Kyley's 3 pens, above, are all different colors?

1. One possible order is black, red, purple.
2. $P(\text{black, red, purple}) = \frac{6}{11} \bullet \frac{4}{10} \bullet \frac{1}{9} = \frac{24}{990}$
3. There are three spots to choose from for the black pen, two spots left for the red pen, and one remaining for the purple pen, for a total of $\binom{3}{1}\binom{2}{1}\binom{1}{1} =$ 6 (that is, ${}_3P_3$) possible orders.
4. $P(\text{all different colors}) = \binom{3}{1}\binom{2}{1}\binom{1}{1} \frac{24}{990} = \frac{144}{990}$

An event must either happen or not happen. Therefore, these two possibilities are complements of each other, making the sum of their probabilities 100%: $P(A) + P(A') = 1$.

Often it is simpler to calculate $P(A')$ than $P(A)$, especially if when A can happen in more ways than A' can. In this case, $P(A)$ can be found by subtracting its complement from 1: $P(A) = 1 - P(A')$. For example, the probability of getting at least one tails out of four coin flips could be calculated by simply $1 - P(0)$ rather than by $P(1) + P(2) + P(3) + P(4)$.

❸ Calculate the probability of at least or at most x out of n occurrences of an event.

1. Count how many ways the event can occur and how many ways its complement can occur to identify whether it will be simpler to calculate the probability of the outcome as stated or of its complement.
2. Calculate the sum of the probabilities identified in step 1 (see ❶).
3. If you calculated the complement, subtract its probability from 1.

❸ Find the probability that out of five 6-sided dice, at least two will roll a 6.

1. The outcome as stated involves the sum of four different probabilities: $P(2) + P(3) + P(4) + P(5)$. The complement is only two: $P(0) + P(1)$.
2. $P(0) = \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} = \frac{3125}{7776}$
   $P(1) = \binom{5}{1} \frac{1}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} = \frac{3125}{7776}$
   $P(0) + P(1) = \frac{3125}{7776} + \frac{3125}{7776} = \frac{6250}{7776}$
3. $P(\geq 2) = \frac{7776}{7776} - \frac{6250}{7776} = \frac{1526}{7776}$