CHAPTER ONE: POLYNOMIALS — Review September 6 ➺ Test September 13

This chapter discusses how to describe and manipulate polynomials, including long division which conceptually is done by the same procedure as long division with numbers. The most fundamental part of the chapter is properties of exponents, which are of great importance to all mathematics and should be practiced until they become automatic.

1-A Terminology — Monday • 8/21

natural • integer • rational • real • complex • irrational • imaginary • term • expression • argument • monomial • degree • constant • linear • quadratic • cubic • polynomial • binomial • trinomial • standard form • coefficient • leading coefficient

❶ Identify a number as natural, integer, rational, real, or complex.
❷ Identify terms and arguments.
❸ Simplify a fraction with multiple terms in the numerator or denominator.
❹ Classify a polynomial in one variable.

1-B Graphs of Polynomials — Thursday • 8/24

❶ Identify the end behavior of the graph of a polynomial.
❷ Sketch a polynomial function.

1-C Properties of Exponents — Monday • 8/28

❶ Simplify an expression using properties of exponents.

1-D Addition, Subtraction, and Multiplication of Polynomials — Wednesday • 8/30

❶ Add or subtract polynomials.
❷ Multiply two polynomials.
❸ Expand a squared binomial.
❹ Multiply more than two polynomials.

1-E Factoring — Friday • 9/1

factoring • common monomial

❶ Factor a trinomial by guessing and checking.
❷ Factor a common monomial out of each term of a polynomial.
❸ Factor a polynomial by grouping.
❹ Factor a perfect square trinomial, a difference of two squares, a difference of two cubes, or a sum of two cubes.
❺ Factor any polynomial.

1-F Division of Polynomials — Wednesday • 9/6

dividend • divisor

❶ Divide polynomials using long division.
❷ Finish factoring a polynomial by dividing by a known factor.

1-A Terminology

Every number is contained in one or more of five number sets: All numbers are complex, some complex numbers are real, some real numbers are rational, some rational numbers are integers, and some integers are natural.

Set	Contents	Examples
$\mathbb{N}$: NATURAL	1, 2, 3, …	8, 21
$\mathbb{Z}$: INTEGER	$\mathbb{N}$, -$\mathbb{N}$, 0	-8, -21, $\mathbb{N}$
$\mathbb{Q}$: RATIONAL	$\mathbb{Z} \div \mathbb{Z}$	$\frac{8}{21}$, -8.21, $\mathbb{Z}$
$\mathbb{R}$: REAL	all numbers on numberline	$\sqrt{21}$, $\sqrt[8]{21}$, log 21, π, e, $\mathbb{Q}$
$\mathbb{C}$: COMPLEX	$\mathbb{R} + \mathbb{R}i$, where $i = \sqrt{-1}$	$8 + 21i$, $8i$, $\mathbb{R}$

IRRATIONAL Numbers are real numbers that are not rational.

IMAGINARY Numbers are complex numbers that are not real. They are also called nonreal numbers.

Subsets of these sets that contain only positive or only negative numbers are denoted with a superscript + or -, such as $\mathbb{R}^+$.

❶ Identify a number as natural, integer, rational, real, or complex.

1. All numbers are complex.
2. If it does not involve an even root of a negative number, it is also real. Otherwise it is imaginary.
3. If it can be written exactly as a fraction of integers, it is also rational. Otherwise it is irrational.
   Keep in mind that all decimals can be written as fractions except those with no pattern or end. For example, $0.27 = \frac{27}{100}$ and $0.272727... = \frac{3}{11}$.
4. If it can be written with no decimal point, fraction bar, or any other symbol other than a negative sign, it is also an integer.
5. If it is a positive integer, it is natural.

❶ Sort the following numbers into the smallest set that includes each: 2, -2, $\sqrt{2}$, $\sqrt{-2}$, -22, 2.2, $2 - \sqrt{-2}$, 2×10^{22}, 2×10^{-22}, π.

1. $\sqrt{-2}$ and $2 - \sqrt{-2}$ involve even (in this case square) roots of negative numbers. They are complex but not real.
2. $\pi \approx 3.1416$ and $\sqrt{2} \approx 1.4142$ are not imaginary but cannot be written as exact decimals. They are real but not rational.
3. 2.2 and 2×10^{-22} can be written as exact fractions, but are not whole numbers. They are rational but not integers.
4. -2 and -22 can be written without fractions, decimals, radicals, etc., but are not positive. They are integers but not natural.
5. 2 and 2×10^{22} are positive whole numbers. They are natural.

A TERM is a constant nonzero value multiplied by a variable value.
An Algebraic EXPRESSION is one or more terms added together.
An ARGUMENT of a Function is an expression input into the function.

❷ Identify terms and arguments.

1. An argument is a value acted on by a function.
2. Each individual term includes no addition or subtraction, but when the terms of an expression are added the result is the whole expression.

❷ Identify the terms and arguments in the expression $5x^2 - \frac{8x}{7} + 9 + 10\sqrt{x - \cos 3x}$.

1. $3x$ is the argument of the cosine function.
2. $3x$ is the only term in this argument.
1. $x - \cos 3x$ is the argument of the square root function.
2. x and $-\cos 3x$ are the terms of this argument, because when added the result is the whole expression $x - \cos 3x$.
2. $5x^2$, $\frac{-8x}{7}$, 9, and $10\sqrt{x - \cos 3x}$ are terms of the expression, because when added the result is the whole expression $5x^2 - \frac{8x}{7} + 9 + 10\sqrt{x - \cos 3x}$. Note that 10, x, $3x$, and $-\cos 3x$ are not terms of the whole expression.

A function and its argument is a single term. An operation on such a term affects the term as a whole, not the argument separately.

❸ Simplify a fraction with multiple terms in the numerator or denominator.

1. Identify the terms of the numerator and the denominator.
2. Identify a factor that divides evenly into all of the terms.
3. Rewrite the fraction with the factor divided out of each term. If it is variable, specify that it cannot be zero.

❸ $\frac{6x^2 - 9x\sqrt{30x}}{6x^2 - 9x}$

1. The terms are $6x^2$, $-9x\sqrt{30x}$, $6x^2$, and $-9x$. $3x$ is an argument of the square root function, not a term of the numerator.
2. $6x^2 \div 3x = 2x$
   $-9x\sqrt{30x} \div 3x = -3\sqrt{30x}$ (not $-3\sqrt{10}$)
   $6x^2 \div 3x = 2x$
   $-9x - 3x = -3$
3. $\frac{2x - 3\sqrt{30x}}{2x - 3}$, $x \neq 0$

Note that $6x^2$ in the numerator and $6x^2$ in the denominator do not cancel each other out because there are other terms.

A MONOMIAL is a nonzero term with a nonnegative integer exponent. The exponent is the DEGREE.

Degree	Name	Example
0	CONSTANT	5 (that is, $5x^0$)
1	LINEAR	$5x$
2	QUADRATIC	$5x^2$
3	CUBIC	$5x^3$
n	n^{th} degree	$5x^{12}$

A POLYNOMIAL is the sum of one or more monomials, after like terms are combined. The degree of a polynomial is the highest degree of its terms.

A BINOMIAL is a polynomial with two terms.

A TRINOMIAL is a polynomial with three terms.

A polynomial with its terms written in order from highest degree to lowest is in STANDARD Form.

A COEFFICIENT is a constant value that is multiplied by a variable value. If a variable value is in a numerator, the coefficient is the rest of the fraction.

The LEADING Coefficient of a Polynomial is the coefficient of the term with the highest degree.

❹ Classify a polynomial in one variable.

1. A single term is a monomial, two terms is a binomial, and three terms is a trinomial.
2. Consider the term with the highest exponent:

   If there is no variable (that is, a variable with an exponent of 0), the polynomial is a constant.

   If there is a variable with no exponent (that is, an exponent of 1), the polynomial is linear

   If the variable is to the second power, the polynomial is quadratic

   If the variable is to the third power, the polynomial is cubic.

   If the variable is to the fourth power or higher, look up what it is called, or simply refer to the polynomial as "n^{th} degree", where n is the exponent.

❹ Write the following polynomials in standard form, classify them, and identify the leading coefficient.

	a) $x + 4x^3$	b) $-15x$	c) $8x^2 - 2x^9 + 3$	d) $2 - \frac{7x^3}{5} + 6x^2 + x$
standard form:	$4x^3 + x$	$-15x$	$-2x^9 + 8x^2 + 3$	$\frac{7x^3}{5} + 6x^2 + x + 2$
classification:	cubic binomial	linear monomial	9^{th} degree trinomial	cubic polynomial
leading coefficient:	4	-15	-2	$\frac{7}{5}$

1-B Graphs of Polynomials

The value (y) of a polynomial function approaches ∞ or -∞ as x approaches ∞ or -∞.

On the right side of the graph, the value of the function has the same sign as the leading coefficient. This is reversed on the left side of the graph if and only if the degree of the polynomial is odd.

❶ Identify the end behavior of the graph of a polynomial.

1. The values on the right side of the graph approach ∞ if the leading coefficient is positive or approach -∞ if the leading coefficient is negative.
2. If the degree of the polynomial is even, repeat step 1 for the left side of the graph. If the degree of the polynomial is odd, do the opposite for the left side of the graph.

❶ $f(x) = -x^3 + 3x - 2$

1. The leading coefficient, -1, is negative, so the values on the right side of the graph approach -∞.
2. The degree, 3, is odd, so the values on the left side do the opposite of the right side and instead approach ∞.

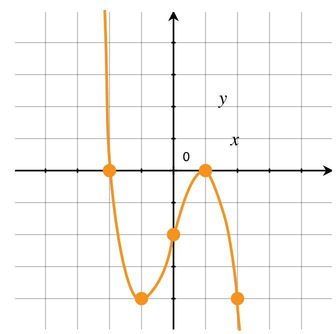
❷ Sketch a polynomial function.

1. Plug in several different values of x to make a table of points $(x, f(x))$.
2. Connect the points in step 1 with a curve.
3. Check which way the points should be trending at the right edge of the graph (see ❶). If they are going in the correct direction, continue the curve in this direction. Otherwise, add more points to the table in step 1.
4. Repeat step 3 for the left side of the graph.

❷ $f(x) = -x^3 + 3x - 2$

1.

x:	-3	-2	-1	0	1	2	3
$f(x)$:	16	0	-4	-2	0	-4	-20



3. The values on the right are decreasing, which is correct.
4. The values on the left are increasing, which is correct.

Note that this method may result in a sketch somewhat different from the actual graph, since the points plotted do not necessarily include the local maximum and minimum points of the function.

1-C Properties of Exponents

The properties below are valid in almost all contexts, including any time a and b are positive or x and y are integers.

Property	Rule	Example
Power of a Product	$(ab)^x = a^x b^x$	$(2x)^3 = 2^3 x^3 = 8x^3$
Power of a Quotient	$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$	$\left(\frac{x}{2}\right)^3 = \frac{x^3}{2^3} = \frac{x^3}{8}$
Power of a Power	$(b^x)^y = b^{xy}$	$(x^5)^3 = x^{15}$
Product of Powers	$b^x b^y = b^{x+y}$	$x^5 x^3 = x^8$
Quotient of Powers	$\frac{b^x}{b^y} = b^{x-y}$	$\frac{x^5}{x^3} = x^2$
Zero Exponent	$b^0 = 1$	$2^0 = 1$
Negative Exponent	$b^{-x} = \frac{1}{b^x}$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

❶ Simplify an expression using properties of exponents.

1. Use the properties above as needed.
2. Reduce numbers if needed.

❶ Simplify $\frac{(2a^4b)^3b^6}{a^{12}bc^2}$, and write it without a fraction. State each property of exponents used.

$(2a^4b)^3 = 2^3(a^4)^3b^3$	Power of a Product	$\frac{8(a^4)^3b^3b^6}{a^{12}bc^2}$
$(a^4)^3 = a^{12}$	Power of a Power	$\frac{8a^{12}b^3b^6}{a^{12}bc^2}$
$b^3b^6 = b^9$	Product of Powers	$\frac{8a^{12}b^9}{a^{12}bc^2}$
$\frac{a^{12}b^9}{a^{12}b} = a^0b^8$	Quotient of Powers	$\frac{8a^0b^8}{c^2}$
$a^0 = 1$	Zero Exponent	$\frac{8(1)b^8}{c^2}$
$\frac{1}{c^2} = c^{-2}$	Negative Exponent	$8b^8c^{-2}$

1-D Addition, Subtraction, and Multiplication of Polynomials

❶ Add or subtract polynomials.

1. Distribute any coefficient, including negatives.
2. Combine like terms.

❶ $5\,(4x^2 + 9x - 3) - (11x - 4)$

1. $(20x^2 + 45x - 15) + (-11x + 4)$
2. $20x^2 + 34x - 11$

❷ Multiply two polynomials.

1. Multiply each term in the first polynomial by each term in the second polynomial.
2. Simplify.
3. Combine like terms.

❷ $(4x^2 - 3x)\,(x + 5)$

1. $(4x^2 \bullet x) + (4x^2 \bullet 5) + (-3x \bullet x) + (-3x \bullet 5)$
2. $4x^3 + 20x^2 - 3x^2 - 15x$
3. $4x^3 + 17x^2 - 15x$

❸ Expand a squared binomial.

Write it as two separate polynomials and multiply, or use the shortcut $(a + b)^2 = a^2 + 2ab + b^2$. Do not leave out the middle term.

❸ $(3x + 10)^2$

incorrect: $(3x)^2 + 10^2 = 9x^2 + 100$

correct: $(3x)^2 + 2(3x)(10) + 10^2 = 9x^2 + 60x + 100$

❹ Multiply more than two polynomials.

1. Multiply the first two polynomials together (see ❷).
2. Multiply the result by the next polynomial.
3. Combine like terms.

❹ $(x + 2)\,(x + 5)\,(x - 10)$

1. $(x^2 + 7x + 10)\,(x - 10)$
2. $(x^3 + 7x^2 + 10x) + (-10x^2 - 70x - 100)$
3. $x^3 - 3x^2 - 60x - 100$

1-E Factoring

FACTORING a Polynomial is writing it as a product of other polynomials.

❶ Factor a trinomial by guessing and checking.

1. Choose two terms that multiply together to equal the first term in the trinomial.
2. Choose two terms that multiply together to equal the last term in the trinomial.
3. Create two binomials from your four terms.
4. Multiply the binomials together to see if you get the desired polynomial. If not, start over with new terms.

❶ $12x^2 - 4x - 5$	retry:
1. $6x$ and $2x$	$6x$ and $2x$
2. 5 and -1	-5 and 1
3. $(6x + 5)$ and $(2x - 1)$	$(6x - 5)$ and $(2x + 1)$
4. $(6x + 5)(2x - 1) = 12x^2 + 4x + 5$ X	$(6x - 5)(2x + 1) = 12x^2 - 4x - 5$ √

A COMMON Monomial is a term that divides evenly into every term in a polynomial.

❷ Factor a common monomial out of each term of a polynomial.

1. Find a monomial that divides evenly into every term in the polynomial.
2. Divide every term by the common monomial.
3. Write the common monomial, followed by the new polynomial in parentheses.
4. Repeat if possible.

❷ $40x^5 - 8x^3 + 20x^2$

1. $4x^2$ divides into all three terms.
2. $(40x^5 - 8x^3 + 20x^2) \div 4x^2 = 10x^3 - 2x + 5$
3. $4x^2(10x^3 - 2x + 5)$

Sometimes one monomial is common to some of the terms in a polynomial, and another monomial is common to the other terms. If factoring these separate monomials out of their respective terms leaves the same quotient each time, this can help in factoring the polynomial.

❸ Factor a polynomial by grouping.

1. Sort the terms of the polynomial into groups such that each group has its own common monomial.
2. Factor a common monomial out of each group.
3. If the quotient of each group is the same, the factors of the polynomial are this quotient and the sum of the common monomials. If not, start over with different groupings or different monomials.

❸ $2x^3 - 8x^2 + 5x - 20$

1. $(2x^3 - 8x^2) + (5x - 20)$
2. $2x^2(x - 4) + 5(x - 4)$
3. $(2x^2 + 5)(x - 4)$

The following special cases are easy to factor when recognized.

Type of Polynomial	Polynomial	Factors
Perfect Square Trinomial	$a^2 + 2ab + b^2$	$(a + b)^2$
Difference of Two Squares	$a^2 - b^2$	$(a + b)\,(a - b)$
Difference of Two Cubes	$a^3 - b^3$	$(a - b)\,(a^2 + ab + b^2)$
Sum of Two Cubes	$a^3 + b^3$	$(a + b)\,(a^2 - ab + b^2)$

❹ Factor a perfect square trinomial, a difference of two squares, a difference of two cubes, or a sum of two cubes.

1. Disregarding negatives, let a be the square root or cube root of the first term and let b be the square root or cube root of the last term.
2. See which of the four polynomials in the second column above matches the polynomial you are factoring.
3. Fill in a and b in the factorization shown in the third column above.
4. For a possible perfect square, give b the same sign as the middle term of the polynomial, and multiply the factors together to see if they do in fact result in the desired polynomial.

❹ Use special cases to factor the following polynomials if possible.

Polynomial	a	b	Type of Polynomial	Factors
a) $x^2 - 100y^2$	x	$10y$	difference of two squares	$(x + 10y)\,(x - 10y)$
b) $x^{10} - 100y^4$	x^5	$10y^2$	difference of two squares	$(x^5 + 10y^2)\,(x^5 - 10y^2)$
c) $x^3 - 1000y^3$	x	$10y$	difference of two cubes	$(x - 10y)\,(x^2 - 10xy + 100y^2)$
d) $27m^3 + 8p^3$	$3m$	$2p$	sum of two cubes	$(3m + 2p)\,(9m^2 - 6mp + 4p^2)$
e) $4x^2 - 20x + 25$	$2x$	-5	perfect square	$(5x - 3)^2$
f) $4x^2 - 10x + 25$	$2x$	-5	not a perfect square, because $(2x - 5)^2 = 4x^2 - 20x + 25$, not $4x^2 - 10x + 25$	

❺ Factor any polynomial.

1. If possible, factor out a common monomial from every term (see ❷).
2. If possible, factor a perfect square, a difference of squares or of cubes, or a sum of cubes (see ❹).
3. If the polynomial is not yet completely factored, try factoring by grouping (see ❸) or by guessing and checking (see ❶).

❺ $80x^7 - 180x^5$

1. $20x^5\,(4x^2 - 9)$
2. $20x^5\,(2x + 3)\,(2x - 3)$

1-F Division of Polynomials

Long division can be used to divide polynomials using the same process as is used to divide numbers. In division, the DIVIDEND is divided by the DIVISOR.

❶ Divide polynomials using long division.

1. Write the problem as long division. Fill in 0's for any missing terms.
2. Divide the first term of the dividend by the first term of the divisor, and write the quotient at the top.
3. Multiply the divisor by the quotient in step 2, and write the product at the bottom of the problem.
4. Subtract the product from step 3 from the like terms of the dividend. The first term should cancel.
5. Bring down the next term in the dividend.
6. Repeat steps 2-5, but using the new expression at the bottom instead of the original dividend.
7. Repeat step 6 until the degree of the difference in step 4 is less than the degree of the divisor.
8. If anything remains of the dividend, write this remainder as a numerator and use the divisor as the denominator.

❶ Divide $2x^5 + 9x^4 + 7x^3 - 8x + 1$ by $x^2 + 5x$.

1. Write the problem, including the term $0x^2$.
2. Write $2x^5 \div x^2 = 2x^3$ at the top.
3. Write $2x^3\,(x^2 + 5x) = 2x^5 + 10x^4$ at the bottom.
4. Write $(2x^5 + 9x^4) - (2x^5 + 10x^4) = -x^4$ at the bottom.
5. Bring down the $7x^3$ from the dividend.
6. Repeat steps 2-5, this time starting with $-x^4 \div x^2$.
7. Continue repeating steps 2-5 until the degree of the difference is lower than 2. (In this case, $292x + 1$ is degree 1.)
8. Add the remainder, which is $\frac{292x+1}{x^2+5x}$, to the answer.

$$
\begin{array}{r l}
 & 2x^3 - x^2 + 12x - 60 + \frac{292x+1}{x^2+5x} \\
(x^2 + 5x) & \overline{)\,2x^5 + 9x^4 + 7x^3 + 0x^2 - 8x + 1} \\
 & \underline{-\,(2x^5 + 10x^4)} \\
 & \quad -x^4 + 7x^3 \\
 & \quad \underline{-\,(-x^4 - 5x^3)} \\
 & \qquad 12x^3 + 0x^2 \\
 & \qquad \underline{-\,(12x^3 + 60x^2)} \\
 & \qquad\quad -60x^2 - 8x \\
 & \qquad\quad \underline{-\,(-60x - 300x)} \\
 & \qquad\qquad 292x + 1
\end{array}
$$

If a divisor divides into a dividend with no remainder, then it is a factor.

❷ Finish factoring a polynomial by dividing by a known factor.

1. Divide the polynomial by the factor.
2. Check that the remainder is 0.
3. Use the result to factor the polynomial completely.

❷ Factor $x^3 + 6x^2 + 5x - 12$, given that $(x + 4)$ is a factor.

1.
$$\begin{array}{r|l} & x^2 + 2x - 3 \\ \hline (x+4) & x^3 + 6x^2 + 5x - 12 \\ & \underline{-(x^3 + 4x^2)} \\ & \quad 2x^2 + 5x \\ & \quad \underline{-(2x^2 + 8x)} \\ & \qquad -3x - 12 \\ & \qquad \underline{-(-3x - 12)} \\ & \qquad\qquad 0 \end{array}$$

2. 0

3. $(x + 4)\,(x^2 + 2x - 3)$

$(x + 4)\,(x + 3)\,(x - 1)$